

# Local Stationarity for Lattice Dynamics in the Harmonic Approximation

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## Abstract

We consider the lattice dynamics in the harmonic approximation for a simple hypercubic lattice with arbitrary unit cell. The initial data are random according to a probability measure which enforces slow spatial variation on the linear scale  $\varepsilon^{-1}$ . We establish two time regimes. For times of order  $\varepsilon^{-\gamma}$ ,  $0 < \gamma < 1$ , locally the measure converges to a Gaussian measure which is space-time stationary with a covariance inherited from the initial (in general, non-Gaussian) measure. For times of order  $\varepsilon^{-1}$  this local space covariance changes in time and is governed by a semiclassical transport equation.

*Key words and phrases:* harmonic crystal, random initial data, covariance matrices, weak convergence of measures, semiclassical transport equation.

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# 1 Introduction

For systems consisting of many interacting “particles” as a rule the slow degrees of freedom are linked to local conservation laws. For example for a classical fluid, mass, momentum, and energy are locally conserved and as a consequence mass, momentum, and energy density are the slow degrees of freedom. Thus if the system starts with some general initial conditions, one expects the fast degrees of freedom to die out rapidly. Then, in a spatial region which on one side contains many particles and on the other side is still small compared to the total extent of the system, thus *locally*, the statistical distribution on phase space is stationary under the dynamics within a good approximation. Since the system has not yet reached global stationarity, there is still a slow motion of the parameters characterizing the states of local stationarity. In our example of a classical fluid, local stationarity coincides with local thermal equilibrium and the local equilibrium parameters, density, momentum, and internal energy, evolve according to the Euler equations of fluid dynamics. For other systems with many particles, in general, it is a difficult task to identify the relevant probability measures stationary in time (and usually also in space).

Such a picture for the dynamics of systems with many particles has theoretical and mathematical support. We refer to [11]. If the dynamics is of Hamiltonian form, the list of worked out examples is rather short. One item on the list is lattice dynamics in the harmonic approximation, which has been investigated in great detail by R.L. Dobrushin and collaborators [2]. We reconsider this model for two reasons.

(i) The first one is on a conceptual level. In phonon physics it is standard practice to use the Wigner function  $W(t, r, \theta)$  as density of phonons with wave number  $\theta$  at location  $r$  and at specified time  $t$ .  $W$  evolves according to the semiclassical transport equation

$$\frac{\partial}{\partial t} W(t, r, \theta) = -\nabla \omega(\theta) \nabla_r W(t, r, \theta), \quad (1.1)$$

$\omega(\theta)$  being the dispersion relation of the harmonic crystal. As we will establish,  $W(t, r, \theta) \delta(\theta - \theta')$  at fixed  $r, t$  encodes the covariance of a Gaussian measure on phase space which is invariant under the lattice dynamics. Thus (1.1) can be understood as the equation governing the motion of the parameters which characterize the locally stationary measures. We believe that in this way the results of Dobrushin *et al.* become more transparent and, in addition, the link to the physics of phonons is provided, see [12] for a more detailed discussion.

(ii) The second reason is technically. In the recent years there has been considerable progress in understanding the long time limit of the harmonic crystal in infinite volume [3]. More precisely one starts with a probability measure  $\mu_0$  which is translation invariant and has some mixing properties. If  $\mu_t$  denotes the time-evolved measure at time  $t$ , then the limit

$$\lim_{t \rightarrow \infty} \mu_t = \mu_\infty, \quad (1.2)$$

is established, where  $\mu_\infty$  is a suitable Gaussian measure with mean zero. It turns out that the techniques for proving (1.2) transcribe to the locally stationary situation. Thereby the conditions in the work of Dobrushin *et al.* are considerably streamlined and the proof is simplified. We also generalize from one to an arbitrary space dimension and from one particle per unit cell to an arbitrary number.

In a recent paper [8], A. Mielke studies the same model and also obtains the semiclassical transport equation (1.1) for the Wigner function. However, Mielke imposes deterministic initial data of slow variation, while we impose random initial data with rather strong mixing properties. Therefore the results are disjoint and so are the techniques for proving them. It is of interest to understand whether a “supertheorem” encompassing both cases has a chance to be valid.

## 2 Lattice dynamics in the harmonic approximation

### 2.1 The model

We consider a Bravais lattice with a unit cell which contains a finite number of atoms. For notational simplicity the Bravais lattice is assumed to be simple hypercubic. Let  $x \in \mathbb{Z}^d$  and let  $u(x)$  be the field of displacements in cell  $x$  from the equilibrium position. If  $u$  is small, we may expand the forces to linear order, which then yields the linear  $n$ -component discrete wave equation

$$\ddot{u}(x, t) = - \sum_{y \in \mathbb{Z}^d} V(x - y) u(y, t), \quad u(x)|_{t=0} = u_0(x), \quad \dot{u}(x)|_{t=0} = v_0(x), \quad x \in \mathbb{Z}^d. \quad (2.1)$$

Here  $u(x, t) = (u_1(x, t), \dots, u_n(x, t))$ ,  $u_0 = (u_{01}(x), \dots, u_{0n}(x)) \in \mathbb{R}^n$  and correspondingly for  $v_0(x)$ . Physically  $n = d \times (\text{number of atoms in the unit cell})$ . Here we take  $n$  to be an arbitrary positive integer.  $V(x)$  is an  $n \times n$  matrix. The dynamics (2.1) is invariant under lattice translations.

Let us denote by  $Y(t) = (Y^0(t), Y^1(t)) = (u(\cdot, t), \dot{u}(\cdot, t))$ ,  $Y_0 = (Y_0^0, Y_0^1) = (u_0(\cdot), v_0(\cdot))$ . Then (2.1) takes the form of an evolution equation

$$\dot{Y}(t) = \mathcal{A}Y(t), \quad t \in \mathbb{R}, \quad Y(0) = Y_0. \quad (2.2)$$

Formally, this is a linear Hamiltonian system, since

$$\mathcal{A}Y = J \begin{pmatrix} \mathcal{V} & 0 \\ 0 & 1 \end{pmatrix} Y = J \nabla H(Y), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.3)$$

with the Hamiltonian functional

$$H(Y) = \frac{1}{2} \langle v, v \rangle + \frac{1}{2} \langle \mathcal{V}u, u \rangle, \quad Y = (u, v), \quad (2.4)$$

where  $\mathcal{V}$  is the convolution operator with the matrix kernel  $V$ , the kinetic energy is given by  $\frac{1}{2} \langle v, v \rangle = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} |v(x)|^2$ , and the potential energy by  $\frac{1}{2} \langle \mathcal{V}u, u \rangle = \frac{1}{2} \sum_{x, y \in \mathbb{Z}^d} u(x) \cdot V(x - y) u(y)$ .

Here “ $\cdot$ ” stands for the scalar product in the Euclidean space  $\mathbb{R}^n$ , resp. in  $\mathbb{R}^d$ .

We assume that the initial datum  $Y_0$  belongs to the phase space  $\mathcal{H}_\alpha$  for some  $\alpha \in \mathbb{R}$ .

**Definition 2.1**  $\mathcal{H}_\alpha$  is the Hilbert space of pairs  $Y = (u, v)$  of  $\mathbb{R}^n$ -valued functions on  $\mathbb{Z}^d$  equipped with the norm

$$\|Y\|_\alpha^2 = \sum_{x \in \mathbb{Z}^d} \left( |u(x)|^2 + |v(x)|^2 \right) (1 + |x|^2)^\alpha < \infty. \quad (2.5)$$

$\mathcal{H}_\alpha$  is equipped with the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathcal{H}_\alpha)$ .

We impose the following conditions on the matrix  $V$ .

**E1** There exist constants  $C, \alpha > 0$  such that  $\|V(z)\| \leq Ce^{-\alpha|z|}$  for  $z \in \mathbb{Z}^d$ ,  $\|V(z)\|$  denoting the matrix norm.

Let  $\hat{V}(\theta)$  be the Fourier transform of  $V(x)$ , with the convention

$$\hat{V}(\theta) = \sum_{z \in \mathbb{Z}^d} V(z) e^{iz \cdot \theta}, \quad \theta \in \mathbb{T}^d, \quad (2.6)$$

$\mathbb{T}^d$  the  $d$ -torus  $\mathbb{R}^d / (2\pi\mathbb{Z})^d$ .

**E2**  $V$  is even, in the sense that  $V(-z) = V(z)^* \in \mathbb{R}$ , for  $z \in \mathbb{Z}^d$ , where  $V^*$  denotes the adjoint of the matrix  $V$  as acting on  $\mathbb{C}^n$ .

Both conditions imply that  $\hat{V}(\theta)$  is a real-analytic Hermitian matrix-valued function in  $\theta \in \mathbb{T}^d$ .

**E3** The matrix  $\hat{V}(\theta)$  is non-negative definite for every  $\theta \in \mathbb{T}^d$ .

Let us define the Hermitian non-negative definite matrix

$$\Omega(\theta) = (\hat{V}(\theta))^{1/2} \geq 0. \quad (2.7)$$

$\Omega(\theta)$  has the eigenvalues  $0 \leq \omega_1(\theta) < \omega_2(\theta) \dots < \omega_s(\theta)$ ,  $s \leq n$  and the corresponding spectral projections  $\Pi_\sigma(\theta)$  with multiplicity  $r_\sigma = \text{tr} \Pi_\sigma(\theta)$ .  $\theta \mapsto \omega_\sigma(\theta)$  is the  $\sigma$ -th band function. There are special points in  $\mathbb{T}^d$ , where the bands cross, which means that  $s$  and  $r_\sigma$  jump to some other value. Away from such crossing points  $s$  and  $r_\sigma$  are independent of  $\theta$ . More precisely one has the following lemma.

**Lemma 2.2** (see [3, Lemma 2.2]). *Let the conditions **E1**, **E2** hold. Then there exists a closed subset  $\mathcal{C}_* \subset \mathbb{T}^d$  such that*

- i) the Lebesgue measure of  $\mathcal{C}_*$  is zero.*
- ii) For every point  $\Theta \in \mathbb{T}^d \setminus \mathcal{C}_*$  there exists a neighborhood  $\mathcal{O}(\Theta)$  such that each band function  $\omega_\sigma(\theta)$  can be chosen as real-analytic function in  $\mathcal{O}(\Theta)$ .*
- iii) The eigenvalue  $\omega_\sigma(\theta)$  has constant multiplicity in  $\mathbb{T}^d \setminus \mathcal{C}_*$ .*
- iv) For  $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$ , the spectral decomposition*

$$\Omega(\theta) = \sum_{\sigma=1}^s \omega_\sigma(\theta) \Pi_\sigma(\theta) \quad (2.8)$$

*holds, where  $\Pi_\sigma(\theta)$  is an orthogonal projection in  $\mathbb{R}^n$ .  $\Pi_\sigma$  is a real-analytic function on  $\mathbb{T}^d \setminus \mathcal{C}_*$ .*

For  $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$  we denote by  $\text{Hess}(\omega_\sigma)$  the matrix of second partial derivatives. Our next condition is the following.

**E4** Let  $D_\sigma(\theta) = \det(\text{Hess}(\omega_\sigma(\theta)))$ . Then  $D_\sigma$  does not vanish identically on  $\mathbb{T}^d \setminus \mathcal{C}_*$ ,  $\sigma = 1, \dots, s$ .

Let

$$\mathcal{C}_0 = \{\theta \in \mathbb{T}^d : \det \hat{V}(\theta) = 0\} \text{ and } \mathcal{C}_\sigma = \{\theta \in \mathbb{T}^d \setminus \mathcal{C}_* : \det(\text{Hess}(\omega_\sigma)) = 0\}, \quad \sigma = 1, \dots, s. \quad (2.9)$$

The following lemma has been proved in [3, Appendix].

**Lemma 2.3** *Let the conditions **E1** - **E4** hold. Then the Lebesgue measure of  $\mathcal{C}_k$  vanishes,  $k = 0, 1, \dots, s$ .*

Our final conditions on  $V$  are the following:

**E5** For each  $\sigma \neq \sigma'$ ,  $\omega_\sigma \pm \omega_{\sigma'}$  does not take a constant value on  $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$ .

This condition holds trivially in case  $n = 1$ .

**E6**  $\|\hat{V}^{-1}(\theta)\| \in L^1(\mathbb{T}^d)$ .

If  $\mathcal{C}_0 = \emptyset$ , then  $\|\hat{V}^{-1}(\theta)\|$  is bounded and **E6** holds trivially.

**Remark 2.4** The conditions **E1** - **E6** are fairly general. In particular they can be checked for the case of nearest neighbor coupling only, for which

$$\langle \mathcal{V}u, u \rangle = \sum_{k=1}^n \sum_{x \in \mathbb{Z}^d} \left( \sum_{i=1}^d \gamma_k |u_k(x + e_i) - u_k(x)|^2 + m_k^2 |u_k(x)|^2 \right), \quad \gamma_k > 0, \quad m_k \geq 0, \quad (2.10)$$

where  $e_i = (\delta_{i1}, \dots, \delta_{id})$ . Then the eigenvalues of  $\hat{V}(\theta)$  are

$$\tilde{\omega}_k(\theta) = \sqrt{2\gamma_1(1 - \cos \theta_1) + \dots + 2\gamma_d(1 - \cos \theta_d) + m_k^2}. \quad (2.11)$$

These eigenvalues still have to be labelled according to magnitude and degeneracy as in Lemma 2.2. Clearly **E1** - **E5** hold. In case all  $m_k > 0$  the set  $\mathcal{C}_0$  is empty and condition **E6** holds automatically. Otherwise, if  $m_k = 0$  for some  $k$ ,  $\mathcal{C}_0 = \{0\}$ . Then **E6** is equivalent to the condition  $\omega_k^{-2}(\theta) \in L^1(\mathbb{T}^d)$ , which holds if  $d \geq 3$ . Therefore, the conditions **E1** - **E6** hold for (2.10) provided either i)  $d \geq 3$ , or ii)  $d = 1, 2$  and all  $m_k > 0$ .

**Proposition 2.5** (see [3, Proposition 2.5]). *Let **E1** and **E2** hold and choose some  $\alpha \in \mathbb{R}$ . Then*

i) *for any  $Y_0 \in \mathcal{H}_\alpha$  there exists a unique solution  $Y(t) \in C(\mathbb{R}, \mathcal{H}_\alpha)$  to the Cauchy problem (2.2).*

ii) *The operator  $U(t) : Y_0 \mapsto Y(t)$  is continuous in  $\mathcal{H}_\alpha$ ,  $\|U(t)Y_0\|_\alpha \leq C(t)\|Y_0\|_\alpha$ .*

## 2.2 Random initial data

We assume that  $Y_0$  is a random function with distribution  $\mu_0$ .

**Definition 2.6**  $\mu_t$  is a Borel probability measure in  $\mathcal{H}_\alpha$  which gives the distribution of  $Y(t)$ ,

$$\mu_t(B) = \mu_0(U(-t)B), \quad B \in \mathcal{B}(\mathcal{H}_\alpha), \quad t \in \mathbb{R}.$$

Expectation with respect to  $\mu_t$  is denoted by  $\mathbb{E}_t$ .

We set  $\mathcal{D} = D \oplus D$  with  $D = C_0(\mathbb{Z}^d) \otimes \mathbb{R}^n$ , where  $C_0(\mathbb{Z}^d)$  denotes a space of real sequences with finite support, and  $\langle Y, \Psi \rangle = \langle Y^0, \Psi^0 \rangle + \langle Y^1, \Psi^1 \rangle$  for  $Y = (Y^0, Y^1) \in \mathcal{H}_\alpha$  and  $\Psi =$

$(\Psi^0, \Psi^1) \in \mathcal{D}$ . For a probability measure  $\mu$  on  $\mathcal{H}_\alpha$  we denote by  $\hat{\mu}$  the characteristic functional (Fourier transform),

$$\hat{\mu}(\Psi) = \int \exp(i\langle Y, \Psi \rangle) \mu(dY), \quad \Psi \in \mathcal{D}.$$

A measure  $\mu$  is called Gaussian of zero mean, if its characteristic functional has the form

$$\hat{\mu}(\Psi) = \exp \left[ -\frac{1}{2} \mathcal{Q}(\Psi, \Psi) \right], \quad \Psi \in \mathcal{D},$$

where  $\mathcal{Q}$  is a real non-negative quadratic form on  $\mathcal{D}$ . A measure  $\mu$  is called translation invariant if  $\mu(T_h B) = \mu(B)$ ,  $B \in \mathcal{B}(\mathcal{H}_\alpha)$ ,  $h \in \mathbb{Z}^d$ , where  $T_h Y(x) = Y(x - h)$ ,  $x \in \mathbb{Z}^d$ .

Let  $O(r)$  denote the set of all pairs of subsets  $\mathcal{A}, \mathcal{B} \subset \mathbb{Z}^d$  at a distance  $\text{dist}(\mathcal{A}, \mathcal{B}) \geq r$  and let  $\sigma(\mathcal{A})$  be the  $\sigma$ -algebra in  $\mathcal{H}_\alpha$  generated by  $Y(x)$  with  $x \in \mathcal{A}$ . Define the Ibragimov-Linnik mixing coefficient of a probability measure  $\mu$  on  $\mathcal{H}_\alpha$  by (cf. [6, Definition 17.2.2])

$$\varphi(r) = \sup_{(\mathcal{A}, \mathcal{B}) \in O(r)} \sup_{\substack{A \in \sigma(\mathcal{A}), B \in \sigma(\mathcal{B}) \\ \mu(B) > 0}} \frac{|\mu(A \cap B) - \mu(A)\mu(B)|}{\mu(B)}. \quad (2.12)$$

**Definition 2.7** A measure  $\mu$  satisfies the strong uniform Ibragimov-Linnik mixing condition if  $\varphi(r) \rightarrow 0$  as  $r \rightarrow \infty$ .

## 3 Main results

### 3.1 Spatially homogeneous initial measure

In this subsection we assume that the initial measure,  $\mu_0$ , is spatially translation invariant with the following properties.

**S1**  $Y_0(x)$  has zero expectation value,

$$\mathbb{E}_0(Y_0(x)) = 0, \quad x \in \mathbb{Z}^d.$$

**S2**  $\mu_0$  has translation invariant correlation matrices, i.e., for  $i, j = 0, 1$ ,

$$Q_0^{ij}(x, x') = \mathbb{E}_0(Y_0^i(x) \otimes Y_0^j(x')) = q_0^{ij}(x - x'), \quad x, x' \in \mathbb{Z}^d. \quad (3.1)$$

Here for  $a, b, c \in \mathbb{C}^n$  we denote by  $a \otimes b$  the linear operator  $(a \otimes b)c = a \sum_{j=1}^n b_j c_j$ .

**S3**  $\mu_0$  has a finite variance and finite mean energy density,

$$e_0 = \mathbb{E}_0(|Y_0^0(x)|^2 + |Y_0^1(x)|^2) = \text{tr } q_0^{00}(0) + \text{tr } q_0^{11}(0) < \infty, \quad x \in \mathbb{Z}^d.$$

**S4**  $\mu_0$  satisfies the strong uniform Ibragimov-Linnik mixing condition with

$$\int_0^\infty r^{d-1} \varphi^{1/2}(r) dr < \infty. \quad (3.2)$$

In [3] we prove the weak convergence of the measures  $\mu_t$  to a limit measure  $\mu_\infty$  on the Hilbert space  $\mathcal{H}_\alpha$  with  $\alpha < -d/2$ , which means

$$\lim_{t \rightarrow \infty} \int f(Y) \mu_t(dY) = \int f(Y) \mu_\infty(dY) \quad (3.3)$$

for all bounded continuous functions  $f$  on  $\mathcal{H}_\alpha$ .  $\mu_\infty$  is a Gaussian measure on  $\mathcal{H}_\alpha$ .

**Theorem 3.1** (see [3]). *Let  $d, n \geq 1$ ,  $\alpha < -d/2$ , and assume that the conditions **E1** - **E6** and **S1** - **S4** hold. Then*

*i) the correlation matrices of the measures  $\mu_t$  converge to a limit, for  $i, j = 0, 1$ ,*

$$Q_t^{ij}(x, x') = \int (Y^i(x) \otimes Y^j(x')) \mu_t(dY) \rightarrow Q_\infty^{ij}(x, x'), \quad t \rightarrow \infty.$$

*ii) The convergence in (3.3) holds.*

*iii) The limit measure  $\mu_\infty$  is a Gaussian measure on  $\mathcal{H}_\alpha$ .*

*iv) The correlation matrix of  $\mu_\infty$  is translation invariant,  $Q_\infty(x, x') = q_\infty(x - x')$ , and has the Fourier transform*

$$\hat{q}_\infty(\theta) = \sum_{\sigma=1}^s \Pi_\sigma(\theta) M_0(\theta) \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C}_*,$$

where  $\Pi_\sigma(\theta)$  is the spectral projection from Lemma 2.2 iv) and

$$M_0(\theta) = \frac{1}{2} (\hat{q}_0(\theta) + C(\theta) \hat{q}_0(\theta) C(\theta)^*)$$

with

$$C(\theta) = \begin{pmatrix} 0 & \Omega(\theta)^{-1} \\ -\Omega(\theta) & 0 \end{pmatrix}. \quad (3.4)$$

*v) The measure  $\mu_\infty$  is time stationary, i.e.  $[U(t)]^* \mu_\infty = \mu_\infty$ ,  $t \in \mathbb{R}$ .*

The projection of the initial covariance  $\hat{q}_0(\theta)$  to the limiting covariance  $\hat{q}_\infty(\theta)$  can be stated more concisely through introducing the complex-valued field

$$a(x) = \frac{1}{\sqrt{2}} (\mathcal{V}^{1/4} u(x) + i \mathcal{V}^{-1/4} v(x)) \in \mathbb{C}^n, \quad x \in \mathbb{Z}^d,$$

with complex conjugate field  $a(x)^*$  and distributional Fourier transform  $\hat{a}(\theta)$ .

Obviously  $\mathbb{E}_t(a(x)) = 0$ . The covariance has two parts. By Theorem 3.1 the  $aa$ -, equivalently the  $a^*a^*$ -, covariance satisfies

$$\lim_{t \rightarrow \infty} \mathbb{E}_t(a(x) \otimes a(x')) = 0.$$

For the  $a^*a$ -covariance we define

$$\mathbb{E}_t(\hat{a}(\theta)^* \otimes \hat{a}(\theta')) = (2\pi)^d \delta(\theta - \theta') W(t, \theta),$$

using the translation invariance of  $\mu_t$ . Note that  $W(t, \theta) \geq 0$ . Then

$$\lim_{t \rightarrow \infty} W(t, \theta) = \sum_{\sigma=1}^s \Pi_\sigma(\theta) W(0, \theta) \Pi_\sigma(\theta). \quad (3.5)$$

### 3.2 Initial measure with slow variation

Let  $\{\mu_0^\varepsilon, \varepsilon > 0\}$  be a family of initial measures. Roughly, in a linear region of size  $\varepsilon^{-1}$ ,  $\varepsilon \ll 1$ ,  $\mu_0^\varepsilon$  looks like the spatially homogeneous initial measure from Section 3.1. However the covariance  $Q_0^{ij}$  depends on the spatial region under consideration, and not only on the difference  $x - x'$ .

To be more precise let us introduce the complex  $2n \times 2n$  matrix-valued function  $\hat{R}$  on  $\mathbb{R}^d \times \mathbb{T}^d$ , through

$$\hat{R}(r, \theta) = \begin{pmatrix} \hat{R}^{00}(r, \theta) & \hat{R}^{01}(r, \theta) \\ \hat{R}^{10}(r, \theta) & \hat{R}^{11}(r, \theta) \end{pmatrix}, \quad r \in \mathbb{R}^d, \quad \theta \in \mathbb{T}^d,$$

with the following properties.

**I1** For every fixed  $r \in \mathbb{R}^d$  and  $i, j = 0, 1$ , the entries of the matrix-valued function  $\hat{R}$  are bounded on  $\mathbb{T}^d$  and the inverse Fourier transform

$$R^{ij}(r, x) = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-i\theta \cdot x} \hat{R}^{ij}(r, \theta) d\theta$$

satisfies the bound

$$|R^{ij}(r, x)| \leq C(1 + |x|)^{-\gamma}, \quad x \in \mathbb{Z}^d, \quad (3.6)$$

where  $C$  is some positive constant,  $\gamma > d$ .

**I2** For every fixed  $r \in \mathbb{R}^d$ , the matrix-valued function  $\hat{R}$  satisfies

$$\hat{R}^{00}(r, \theta) \geq 0, \quad \hat{R}^{11}(r, \theta) \geq 0 \quad (3.7)$$

$$\hat{R}^{01}(r, \theta) = \hat{R}^{10}(r, \theta)^*, \quad \theta \in \mathbb{T}^d.$$

**I3** For every fixed  $r \in \mathbb{R}^d$  and  $\theta \in \mathbb{T}^d$ , the matrix  $\hat{R}(r, \theta)$  is non-negative definite.

**I4** For every  $\theta \in \mathbb{T}^d$ ,  $\hat{R}^{ij}(\cdot, \theta)$ ,  $i, j = 0, 1$ , are  $C^1$  functions and the function

$$r \rightarrow \sup_{\theta \in \mathbb{T}^d} \max_{i,j=0,1} (|\hat{R}^{ij}(r, \theta)|, |\nabla_r \hat{R}^{ij}(r, \theta)|)$$

is bounded uniformly on bounded sets.

Let  $\mathbb{E}_0^\varepsilon$  stand for expectation w.r.t. the measure  $\mu_0^\varepsilon$ . We assume that

$$\mathbb{E}_0^\varepsilon(Y^j(x)) = 0 \quad (3.8)$$

and define the covariance

$$Q_\varepsilon^{ij}(x, x') = \mathbb{E}_0^\varepsilon(Y^i(x) \otimes Y^j(x')), \quad x, x' \in \mathbb{Z}^d, \quad i, j = 0, 1.$$

**Definition 3.2** We call a family of measures  $\{\mu_0^\varepsilon, \varepsilon > 0\}$  a family of slow variation for  $R$  if  $\{Q_\varepsilon^{ij}(x, x'), \varepsilon > 0\}$  satisfies the conditions **V1** - **V4** listed below.

**V1** For any  $\varepsilon > 0$  there exists an even integer  $N_\varepsilon$  such that

i) for all  $M \in \mathbb{R}^d$  and  $x, x' \in I_M$ ,

$$|Q_\varepsilon^{ij}(x, x') - R^{ij}(\varepsilon M, x - x')| \leq C \min[(1 + |x - x'|)^{-\gamma}, \varepsilon N_\varepsilon], \quad (3.9)$$

where  $C, \gamma$  are the constants from (3.6), and  $I_M$  is the cube centered at the point  $M$  with edge length  $N_\varepsilon$ ,

$$I_M = \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : |x_j - M_j| \leq N_\varepsilon/2, M = (M_1, \dots, M_d)\}. \quad (3.10)$$

ii)  $N_\varepsilon \sim \varepsilon^{-\beta}$  as  $\varepsilon \rightarrow 0$ , with some  $\beta \in (1/2, 1)$ .

**V2** For any  $\varepsilon > 0$  and all  $x, x' \in \mathbb{Z}^d$ ,  $i, j = 0, 1$ ,

$$|Q_\varepsilon^{ij}(x, x')| \leq C(1 + |x - x'|)^{-\gamma}$$

with constants  $C, \gamma$  as in (3.6).

**V3** For any  $\varepsilon > 0$  and any  $\Psi_1, \Psi_2 \in \mathcal{D}$  with  $\text{dist}(\text{supp } \Psi_1, \text{supp } \Psi_2) \geq \rho > 0$  there exist constants  $C > 0$  and  $\kappa \in (0, 1)$  such that

$$|\mathbb{E}_0^\varepsilon(e^{i\langle Y, \Psi_1 \rangle} e^{i\langle Y, \Psi_2 \rangle}) - \mathbb{E}_0^\varepsilon(e^{i\langle Y, \Psi_1 \rangle}) \mathbb{E}_0^\varepsilon(e^{i\langle Y, \Psi_2 \rangle})| \leq C(1 + \rho)^{-\kappa}.$$

**V4** For correlation functions of the fourth order

$$M_\varepsilon^{(4)}(x^1, x^2, x^3, x^4) = \mathbb{E}_0^\varepsilon(Y(x^1) \otimes Y(x^2) \otimes Y(x^3) \otimes Y(x^4)), \quad x^1, \dots, x^4 \in \mathbb{Z}^d,$$

we require that

$$|M_\varepsilon^{(4)}(x^1, x^2, x^3, x^4)| \leq C \sum_{(i_1, i_2, i_3, i_4) \in P\{1, 2, 3, 4\}} (1 + |x^{i_1} - x^{i_2}|)^{-\gamma} (1 + |x^{i_3} - x^{i_4}|)^{-\gamma},$$

where  $P\{1, 2, 3, 4\}$  is a permutation of the numbers 1, 2, 3, 4, and  $\gamma > d$ .

**Definition 3.3** i)  $\mu_t^\varepsilon$  is a Borel probability measure in  $\mathcal{H}_\alpha$  which gives the joint distribution of  $Y(t)$ ,

$$\mu_t^\varepsilon(B) = \mu_0^\varepsilon(U(-t)B), \quad B \in \mathcal{B}(\mathcal{H}_\alpha), \quad t \in \mathbb{R}.$$

ii) The correlation functions of the measure  $\mu_t^\varepsilon$  are defined by

$$Q_{\varepsilon, t}^{ij}(x, y) = \int Y^i(x) \otimes Y^j(y) \mu_t^\varepsilon(dY) = \mathbb{E}_0^\varepsilon(Y^i(x, t) \otimes Y^j(y, t)), \quad i, j = 0, 1, \quad x, y \in \mathbb{Z}^d.$$

Here  $Y^i(x, t)$  are the components of the solution  $Y(t) = (Y^0(\cdot, t), Y^1(\cdot, t))$ .

### 3.3 Covariance in the kinetic scaling limit

The family  $\mu_0^\varepsilon$ ,  $\varepsilon > 0$ , of initial measures has slow spatial variation on scale  $\varepsilon^{-1}$  and for long times, roughly of order  $\varepsilon^{-\gamma}$ ,  $0 < \gamma < 1$ , in essence Theorem 3.1 applies locally, which implies that locally the projected measure is attained. This measure is then almost invariant under the time evolution. Thus one needs a time span of order  $\tau/\varepsilon$ ,  $\tau \neq 0$ , to see changes in the projected part of the covariance.

To state a precise result we introduce the scaled  $n \times n$  Wigner matrix through

$$W^\varepsilon(\tau; r, \theta) = \sum_{y \in \mathbb{Z}^d} e^{i\theta \cdot y} \mathbb{E}_{\tau/\varepsilon}^\varepsilon \left( a^*([\varepsilon^{-1}r + y/2]) \otimes a([\varepsilon^{-1}r - y/2]) \right). \quad (3.11)$$

By our assumptions on  $\mu_0^\varepsilon$ , the following limit exists

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} W^\varepsilon(0; r, \theta) &= \frac{1}{2} \left( \Omega^{1/2} \hat{R}^{00}(r, \theta) \Omega^{1/2} + \Omega^{-1/2} \hat{R}^{11}(r, \theta) \Omega^{-1/2} \right. \\ &\quad \left. + i\Omega^{1/2} \hat{R}^{01}(r, \theta) \Omega^{-1/2} - i\Omega^{-1/2} \hat{R}^{10}(r, \theta) \Omega^{1/2} \right) \\ &= W(0; r, \theta). \end{aligned}$$

We also define the projected initial Wigner matrix, compare with (3.5),

$$W^p(r, \theta) = \sum_{\sigma=1}^s \Pi_\sigma(\theta) W(0; r, \theta) \Pi_\sigma(\theta) \quad (3.12)$$

and its time evolution

$$W^p(\tau; r, \theta) = \sum_{\sigma=1}^s \Pi_\sigma(\theta) W(0; r - \tau \nabla \omega_\sigma(\theta), \theta) \Pi_\sigma(\theta). \quad (3.13)$$

**Theorem 3.4** *Let the conditions **V1** - **V2** and **E1** - **E6** hold. Then for any  $r \in \mathbb{R}^d$  and  $\tau \neq 0$  the following limit exists in the sense of distributions,*

$$\lim_{\varepsilon \rightarrow 0} W^\varepsilon(\tau; r, \theta) = W^p(\tau; r, \theta). \quad (3.14)$$

*In addition, for the remaining part of the covariance,*

$$\lim_{\varepsilon \rightarrow 0} \sum_{y \in \mathbb{Z}^d} e^{i\theta \cdot y} \mathbb{E}_{\tau/\varepsilon}^\varepsilon \left( a([\varepsilon^{-1}r + y/2]) \otimes a([\varepsilon^{-1}r - y/2]) \right) = 0. \quad (3.15)$$

We remark that in the  $\sigma$ -th band the Wigner function evolves according to the transport equation

$$\frac{\partial}{\partial t} f_t(r, \theta) + \nabla \omega_\sigma(\theta) \cdot \nabla_r f_t(r, \theta) = 0, \quad (3.16)$$

where the initial conditions are given by the initial Wigner matrix projected onto the  $\sigma$ -th band.

The conditions **V1** and **V2** on the initial measure are written in position space. Therefore it is natural to prove the limiting covariance first in position space, which will be stated in Theorem 4.1. From it we deduce the limiting Wigner function of Theorem 3.4.

### 3.4 Local stationarity

So far we studied only the covariance. A more detailed statistical information is provided by considering the random field  $Y$  at the kinetic time  $\tau/\varepsilon$ ,  $\tau \neq 0$ , and close to the spatial point  $[r/\varepsilon] \in \mathbb{Z}^d$ . For this purpose let  $T_h$ ,  $h \in \mathbb{Z}^d$ , be the group of space translations. The measure at  $r/\varepsilon$  is then defined through

$$\mu_{\tau/\varepsilon, r}^\varepsilon = T_{-[r/\varepsilon]} \mu_{\tau/\varepsilon}^\varepsilon. \quad (3.17)$$

**Theorem 3.5** *Let the conditions **V1** - **V4** and **E1** - **E6** hold. Then for  $\tau \neq 0$ , in the sense of weak convergence on  $\mathcal{H}_\alpha$ ,*

$$\lim_{\varepsilon \rightarrow 0} \mu_{\tau/\varepsilon, r}^\varepsilon = \mu_{\tau, r}^G. \quad (3.18)$$

$\mu_{\tau, r}^G$  is a Gaussian measure on  $\mathcal{H}_\alpha$ , which is invariant under the space translations  $T_h$  and time translation  $U(t)$ .  $\mu_{\tau, r}^G$  has mean zero and covariance

$$q_{\tau, r}^{ij}(x - x') = \mathbb{E}_{\tau, r}^G(Y^i(x) \otimes Y^j(x')),$$

expectation with respect to  $\mu_{\tau, r}^G$ . The covariance is determined through  $W^P(\tau; r, \theta)$  as

$$\Omega(\theta) \hat{q}_{\tau, r}^{00}(\theta) = \Omega(\theta)^{-1} \hat{q}_{\tau, r}^{11}(\theta) = \frac{1}{2} (W^P(\tau; r, \theta) + W^P(\tau, -\theta)^*) \quad (3.19)$$

and

$$\hat{q}_{\tau, r}^{01}(\theta) = -\hat{q}_{\tau, r}^{10}(\theta) = -\frac{i}{2} (W^P(\tau; r, \theta) - W^P(\tau; r, -\theta)^*). \quad (3.20)$$

We conclude that close to  $r/\varepsilon$  in space and close to  $\tau/\varepsilon$  in time the random field  $Y^j(x, t)$  is a stationary Gaussian field. Its distribution at fixed local time  $t$  is given by  $\mu_{\tau, r}^G$  while in time it evolves deterministically according  $U(t)$ . In this sense locally in space and time the random field is stationary with statistics determined through the Wigner matrix at  $(r, \tau)$  and the microscopic dynamics, compare with (3.19), (3.20).

## 4 Convergence of correlation functions

At first we introduce the matrix  $q_{\tau, r}(x)$ . In Fourier space,

$$\hat{q}_{\tau, r}(\theta) = \sum_{\sigma=1}^s \Pi_\sigma(\theta) (\mathbf{M}_+^\sigma(\tau; r, \theta) + \mathbf{M}_-^\sigma(\tau; r, \theta)) \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C}_*, \quad (4.1)$$

where  $\Pi_\sigma(\theta)$  is the spectral projection introduced in Lemma 2.2 *iv*),

$$\begin{aligned} \mathbf{M}_+^\sigma(\tau; r, \theta) &= \frac{1}{2} (\mathbf{R}_+^\sigma(\tau; r, \theta) + C(\theta) \mathbf{R}_+^\sigma(\tau; r, \theta) C^*(\theta)), \\ \mathbf{M}_-^\sigma(\tau; r, \theta) &= \frac{1}{2} (C(\theta) \mathbf{R}_-^\sigma(\tau; r, \theta) - \mathbf{R}_-^\sigma(\tau; r, \theta) C^*(\theta)), \end{aligned} \quad (4.2)$$

with matrix  $C(\theta)$  as in (3.4) and

$$\mathbf{R}_\pm^\sigma(\tau; r, \theta) = \frac{1}{2} (\hat{R}(r + \nabla \omega_\sigma(\theta) \tau, \theta) \pm \hat{R}(r - \nabla \omega_\sigma(\theta) \tau, \theta)). \quad (4.3)$$

**Theorem 4.1** *Let the conditions **V1** - **V2** and **E1** - **E6** hold. Then for any  $r \in \mathbb{R}^d$ ,  $x, y \in \mathbb{Z}^d$ ,  $\tau \neq 0$  the correlation functions of measures  $\mu_{\tau/\varepsilon, r}^\varepsilon$  converge to a limit,*

$$\lim_{\varepsilon \rightarrow 0} Q_{\varepsilon, \tau/\varepsilon}^{ij}([r/\varepsilon] + x, [r/\varepsilon] + y) = q_{\tau, r}^{ij}(x - y). \quad (4.4)$$

We outline the strategy of the proof. For the proof we use the cutting strategy from [3] combined with some techniques from [2], where Theorem 4.1 has proved for the case  $d = n = 1$  (see [2, Theorem 3.1]). Note that in [2] it is assumed the stronger conditions on matrix  $V$  than **E3**, **E4**, namely,  $\omega(\theta) > 0$ , and the set

$$\{\theta \in [-\pi, \pi] : \omega''(\theta) = \omega'''(\theta) = 0\}$$

is empty. Under these conditions, in [2] the uniform asymptotics of the Green function is proved,

$$\sup_{x \in \mathbb{Z}^d} |\mathcal{G}_t(x)| \leq C(1 + |t|)^{-1/3}. \quad (4.5)$$

This bound plays an important role in the proof of [2]. However, if  $n > 1$ , then  $\omega_s$  may be non-smooth because of band crossing, and if  $d > 1$ , the set where the Hessian vanishes does not consist of isolated points. Therefore a strong estimate as (4.5) is unlikely to be valid, in general. To cope with such a situation, we split  $\mathcal{G}_t(x)$  into two summands:  $\mathcal{G}_t(x) = \mathcal{G}_t^f(x) + \mathcal{G}_t^g(x)$ , where  $\mathcal{G}_t^f(x)$  has a support in the neighborhood of a “critical set”  $\mathcal{C} \subset \mathbb{T}^d$ , and  $\mathcal{G}_t^g(x)$  vanishes in the neighborhood of  $\mathcal{C}$ . The set  $\mathcal{C}$  includes all points  $\theta \in \mathbb{T}^d$  either with a degenerate Hessian of  $\omega_\sigma(\theta)$ , or with non-smooth  $\omega_\sigma(\theta)$  (see Definition 4.11). We show that the contribution of  $\mathcal{G}_t^f(x)$  is negligible uniformly in  $t$  (see (4.15)). Hence, it allows us to represent correlations functions  $Q_{\varepsilon, \tau/\varepsilon}$  in the form:  $Q_{\varepsilon, \tau/\varepsilon} = Q_{\varepsilon, \tau/\varepsilon}^g + Q_{\varepsilon, \tau/\varepsilon}^r$ , such that

$$Q_{\varepsilon, \tau/\varepsilon}^g(x, y) = \sum_{x', y' \in \mathbb{Z}^d} \mathcal{G}_t^g(x - x') Q_\varepsilon(x', y') \mathcal{G}_t^g(y - y')^*.$$

For the remainder  $Q_{\varepsilon, \tau/\varepsilon}^r = Q_{\varepsilon, \tau/\varepsilon} - Q_{\varepsilon, \tau/\varepsilon}^g$  we prove that  $Q_{\varepsilon, \tau/\varepsilon}^r(x, y) = o(1)$  uniformly in  $\tau \neq 0$ ,  $\varepsilon > 0$  and  $x, y \in \mathbb{Z}^d$ . The last fact follows from two key observations: i)  $\text{mes } \mathcal{C} = 0$  (Lemma 2.2) and ii) the correlation quadratic form is continuous in  $\ell^2$ , see Corollary 4.3. Up to this point we apply the “cutting strategy” from [3, 4]. Finally, in Section 4.3 we prove that  $Q_{\varepsilon, \tau/\varepsilon}^g([r/\varepsilon] + x, [r/\varepsilon] + y)$  converges to a limit as  $\varepsilon \rightarrow 0$ , using the techniques of [2]. In addition, the asymptotics of  $\mathcal{G}_t^g(x)$ , (see Lemma 4.5) of the form  $\mathcal{G}_t^g(x) \sim (1 + |t|)^{-d/2}$  plays the important role, since it replaces the asymptotics (4.5) and also simplifies some steps of the proof of [2].

## 4.1 Bounds for initial covariance

**Definition 4.2** *By  $\ell^p \equiv \ell^p(\mathbb{Z}^d) \otimes \mathbb{R}^n$ ,  $p \geq 1$ ,  $n \geq 1$ , we denote the space of sequences  $f(x) = (f_1(x), \dots, f_n(x))$  endowed with norm  $\|f\|_p = \left( \sum_{x \in \mathbb{Z}^d} |f(x)|^p \right)^{1/p}$ .*

**Lemma 4.3** *Let condition **V2** hold. Then for  $i, j = 0, 1$ , the following bounds hold*

$$\begin{aligned} \sum_{y \in \mathbb{Z}^d} |Q_\varepsilon^{ij}(x, y)| &\leq C < \infty \quad \text{for all } x \in \mathbb{Z}^d, \\ \sum_{x \in \mathbb{Z}^d} |Q_\varepsilon^{ij}(x, y)| &\leq C < \infty \quad \text{for all } y \in \mathbb{Z}^d. \end{aligned}$$

Here the constant  $C$  does not depend on  $x, y \in \mathbb{Z}^d$  and  $\varepsilon > 0$ .

**Corollary 4.4** *Lemma 4.3 implies, by the Shur lemma, that for any  $\Phi, \Psi \in \ell^2$  the following bound holds:*

$$|\langle Q_\varepsilon(x, y), \Phi(x) \otimes \Psi(y) \rangle| \leq C \|\Phi\|_2 \|\Psi\|_2,$$

where a constant  $C$  does not depend on  $\varepsilon > 0$ .

## 4.2 Stationary phase method

Applying Fourier transform to (2.2) we obtain

$$\dot{\hat{Y}}(t) = \hat{\mathcal{A}}(\theta) \hat{Y}(t), \quad t \in \mathbb{R}, \quad \hat{Y}(0) = \hat{Y}_0. \quad (4.6)$$

Here we denote

$$\hat{\mathcal{A}}(\theta) = \begin{pmatrix} 0 & 1 \\ -\hat{V}(\theta) & 0 \end{pmatrix}, \quad \theta \in \mathbb{T}^d. \quad (4.7)$$

The solution to (2.2) admits the representation

$$Y(x, t) = \sum_{y \in \mathbb{Z}^d} \mathcal{G}_t(x - y) Y_0(y), \quad (4.8)$$

where the Green function  $\mathcal{G}_t(x)$  has the form

$$\mathcal{G}_t(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-ix \cdot \theta} \exp(\hat{\mathcal{A}}(\theta)t) d\theta.$$

Note that

$$\hat{\mathcal{G}}_t(\theta) = \begin{pmatrix} \cos \Omega t & \sin \Omega t \, \Omega^{-1} \\ -\sin \Omega t \, \Omega & \cos \Omega t \end{pmatrix}, \quad (4.9)$$

where  $\Omega = \Omega(\theta)$  is the Hermitian matrix defined by (2.7). Hence, we can rewrite  $\mathcal{G}_t(x)$  in the form

$$\mathcal{G}_t(x) = \sum_{\pm, \sigma=1}^s \int_{\mathbb{T}^d} e^{-ix \cdot \theta} e^{\pm i \omega_\sigma(\theta)t} a_\sigma^\pm(\theta) d\theta. \quad (4.10)$$

We are going to apply the stationary phase arguments to the integral (4.10) which require a smoothness in  $\theta$ . Then we have to choose certain smooth branches of the functions  $a_\sigma^\pm(\theta)$  and  $\omega_\sigma(\theta)$  and cut off all singularities. First, we introduce the *critical set* as

$$\mathcal{C} = \mathcal{C}_* \bigcup_{\sigma=1}^s \mathcal{C}_\sigma \bigcup_{i=1}^d \bigcup_{\sigma=1}^s \left\{ \theta \in \mathbb{T}^d : \frac{\partial^2 \omega_\sigma(\theta)}{\partial \theta_i^2} = 0 \right\}, \quad (4.11)$$

with  $\mathcal{C}_*$  as in Lemma 2.2 and sets  $\mathcal{C}_0$  and  $\mathcal{C}_\sigma$  defined by (2.9). Obviously  $\text{mes } \mathcal{C} = 0$ . Secondly, fix an  $\delta > 0$  and choose a finite partition of unity

$$f(\theta) + g(\theta) = 1, \quad g(\theta) = \sum_{m=1}^M g_m(\theta), \quad \theta \in \mathbb{T}^d,$$

where  $f, g_m$  are non-negative functions from  $C_0^\infty(\mathbb{T}^d)$ , and

$$\text{supp } f \subset \{\theta \in \mathbb{T}^d : \text{dist}(\theta, \mathcal{C}) < \delta\}, \quad \text{supp } g_m \subset \{\theta \in \mathbb{T}^d : \text{dist}(\theta, \mathcal{C}) \geq \delta/2\}. \quad (4.12)$$

Then we represent  $\mathcal{G}_t(x)$  in the form  $\mathcal{G}_t(x) = \mathcal{G}_t^f(x) + \mathcal{G}_t^g(x)$ , where

$$\mathcal{G}_t^f(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-ix \cdot \theta} f(\theta) \hat{\mathcal{G}}_t(\theta) d\theta, \quad (4.13)$$

$$\mathcal{G}_t^g(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-ix \cdot \theta} g(\theta) \hat{\mathcal{G}}_t(\theta) d\theta = \sum_{\pm, \sigma=1}^s \sum_{m=1}^M \int_{\mathbb{T}^d} g_m(\theta) e^{-ix \cdot \theta \pm i\omega_\sigma(\theta)t} a_\sigma^\pm(\theta) d\theta. \quad (4.14)$$

By Lemma 2.2 and the compactness arguments, we can choose the supports of  $g_m$  so small that the eigenvalues  $\omega_\sigma(\theta)$  and the amplitudes  $a_\sigma^\pm(\theta)$  are real-analytic functions inside the  $\text{supp } g_m$  for every  $m$ . (We do not label the functions by the index  $m$  to not overburden the notations.) For the function  $\mathcal{G}_t^f(x)$ , the Parseval identity, (4.9), and condition **E6** imply

$$\|\mathcal{G}_t^f(\cdot)\|_2^2 = C \int_{\mathbb{T}^d} |\hat{\mathcal{G}}_t(\theta)|^2 |f(\theta)|^2 d\theta \leq C \int_{\text{dist}(\theta, \mathcal{C}) < \delta} |\hat{\mathcal{G}}_t(\theta)|^2 d\theta \rightarrow 0 \quad \text{as } \delta \rightarrow 0, \quad (4.15)$$

uniformly in  $t \in \mathbb{R}$ . For the function  $\mathcal{G}_t^g(x)$  the following lemma holds.

**Lemma 4.5** *Let conditions **E1** - **E4** and **E6** hold. Then*

$$i) \quad \sup_{x \in \mathbb{Z}^d} |\mathcal{G}_t^g(x)| \leq C t^{-d/2}. \quad (4.16)$$

ii) *For any  $p > 0$  there exist  $C_p, \gamma_g > 0$  such that*

$$|\mathcal{G}_t^g(x)| \leq C_p (|t| + |x| + 1)^{-p}, \quad |x| \geq \gamma_g t. \quad (4.17)$$

**Proof** Consider  $\mathcal{G}_t^g(x)$  along each ray  $x = vt$  with arbitrary  $v \in \mathbb{R}^d$ . By (4.14), one obtains

$$\mathcal{G}_t^g(vt) = \sum_{m=1}^M \sum_{\pm, \sigma=1}^s \int_{\mathbb{T}^d} g_m(\theta) e^{-i(\theta \cdot v \mp \omega_\sigma(\theta))t} a_\sigma^\pm(\theta) d\theta.$$

This is a sum of oscillatory integrals with the phase functions  $\phi_\sigma^\pm(\theta) = \theta \cdot v \mp \omega_\sigma(\theta)$ . Since  $\omega_\sigma(\theta)$  is real-analytic, each function  $\phi_\sigma^\pm$  has no more than a finite number of stationary points  $\theta \in \text{supp } g$ , which are solutions to the equation  $v = \pm \nabla \omega_\sigma(\theta)$ . The stationary points are non-degenerate for  $\theta \in \text{supp } g_m$ , by (4.11) and (4.12), since

$$\det\left(\frac{\partial^2 \phi_\sigma^\pm}{\partial \theta_i \partial \theta_j}\right) = \mp D_\sigma(\theta) \neq 0, \quad \theta \in \text{supp } g_m.$$

Therefore,  $\mathcal{G}_t^g(vt) = \mathcal{O}(t^{-d/2})$  according to the standard stationary phase method [7, 10]. This implies the bounds (4.16) in each cone  $|x| \leq ct$  with any finite  $c$ .

Further, denote by  $\bar{v}_g = \max_m \max_{\sigma=1, \dots, s} \max_{\theta \in \text{supp } g_m} |\nabla \omega_\sigma(\theta)|$ . Then for  $|v| > \bar{v}_g$  the stationary points do not exist on the  $\text{supp } g$ . Hence, the integration by parts as in [10] yields  $\mathcal{G}_t^g(vt) = \mathcal{O}(t^{-p})$  for any  $p > 0$ . On the other hand, the integration by parts (see (4.13)) implies the similar bound  $\mathcal{G}_t^g(x) = \mathcal{O}((t/|x|)^l)$  for any  $l > 0$ . Therefore, (4.17) follows with any  $\gamma_g > \bar{v}_g$ , which means that the bounds (4.16) hold everywhere.  $\blacksquare$

### 4.3 Proof of Theorem 4.1

**Proof** *Step i*). The representation (4.8) gives

$$Q_{\varepsilon, t}(x, y) = \mathbb{E}_0^\varepsilon(Y(x, t) \otimes Y(y, t)) = \sum_{x', y' \in \mathbb{Z}^d} \mathcal{G}_t(x - x') Q_\varepsilon(x', y') \mathcal{G}_t(y - y')^* \quad (4.18)$$

for any  $t \in \mathbb{R}^1$ . Corollary 4.4 and (4.15) imply

$$Q_{\varepsilon, t}(x, y) = \sum_{x', y' \in \mathbb{Z}^d} \mathcal{G}_t^g(x - x') Q_\varepsilon(x', y') \mathcal{G}_t^g(y - y')^* + o(1),$$

where  $o(1) \rightarrow 0$  as  $\delta \rightarrow 0$  uniformly in  $t \in \mathbb{R}$  and  $x, y \in \mathbb{Z}^d$ . In particular, setting  $t = \tau/\varepsilon$ ,  $x = [r/\varepsilon] + l$  and  $y = [r/\varepsilon]$  we get

$$\begin{aligned} Q_{\varepsilon, \tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon]) &= \sum_{x', y' \in \mathbb{Z}^d} \mathcal{G}_{\tau/\varepsilon}^g([r/\varepsilon] + l - x') Q_\varepsilon(x', y') \mathcal{G}_{\tau/\varepsilon}^g([r/\varepsilon] - y')^* + o(1) \\ &= \sum_{x', y' \in \mathbb{Z}^d} \mathcal{G}_{\tau/\varepsilon}^g(l + x') Q_\varepsilon([r/\varepsilon] - x', [r/\varepsilon] - y') \mathcal{G}_{\tau/\varepsilon}^g(y')^* + o(1). \end{aligned}$$

Let  $c = \gamma_g + |l|$ . Then Lemma 4.5, ii) and condition **V2** imply

$$\begin{aligned} Q_{\varepsilon, \tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon]) &= \sum_{x', y' \in [-c\tau/\varepsilon, c\tau/\varepsilon]^d \cap \mathbb{Z}^d} \mathcal{G}_{\tau/\varepsilon}^g(l + x') Q_\varepsilon([r/\varepsilon] - x', [r/\varepsilon] - y') \mathcal{G}_{\tau/\varepsilon}^g(y')^* \\ &\quad + r_1(\varepsilon, \tau) + o(1), \end{aligned}$$

where  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-p} r_1(\varepsilon, \tau) = 0$  for any  $p > 0$  and  $\tau \in \mathbb{R}^1$ .

*Step ii*). We divide the cube  $[-c\tau/\varepsilon, c\tau/\varepsilon]^d$  onto the cubes  $I_{nN_\varepsilon}$  (see (3.10)),

$$[-c\tau/\varepsilon, c\tau/\varepsilon]^d \subset \bigcup_{n \in J} I_{nN_\varepsilon},$$

where  $J = \{n = (n_1, \dots, n_d) \in \mathbb{Z}^d, |n_j| \leq [c\tau/(\varepsilon N_\varepsilon)] + 1\}$ . Then

$$\begin{aligned} Q_{\varepsilon, \tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon]) &= \sum_{m, n \in J} \sum_{\substack{x' \in I_{mN_\varepsilon} \\ y' \in I_{nN_\varepsilon}}} \mathcal{G}_{\tau/\varepsilon}^g(l + x') Q_\varepsilon([r/\varepsilon] - x', [r/\varepsilon] - y') \mathcal{G}_{\tau/\varepsilon}^g(y')^* \\ &\quad + r_1(\varepsilon, \tau) + o(1) \\ &= \sum_{m \in J} \sum_{x', y' \in I_{mN_\varepsilon}} \mathcal{G}_{\tau/\varepsilon}^g(l + x') Q_\varepsilon([r/\varepsilon] - x', [r/\varepsilon] - y') \mathcal{G}_{\tau/\varepsilon}^g(y')^* \\ &\quad + r_2(\varepsilon, \tau) + r_1(\varepsilon, \tau) + o(1), \end{aligned}$$

where

$$r_2(\varepsilon, \tau) = \sum_{\substack{m, n \in J, m \neq n \\ x' \in I_{mN_\varepsilon}, y' \in I_{nN_\varepsilon}}} \mathcal{G}_{\tau/\varepsilon}^g(l + x') Q_\varepsilon([r/\varepsilon] - x', [r/\varepsilon] - y') \mathcal{G}_{\tau/\varepsilon}^g(y')^*. \quad (4.19)$$

Now we prove that

$$r_2(\varepsilon, \tau) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 \quad (4.20)$$

for any  $\tau \in \mathbb{R}^1$ . Indeed, we divide the sum in the RHS of (4.19) onto two sums  $S_1$  and  $S_2$ , where the first sum  $S_1$  is taken over all  $x' \in I_{mN_\varepsilon}$  and  $y' \in I_{nN_\varepsilon}$  and  $m, n \in J$  such that  $\exists j \in \{1, \dots, d\} : |m_j - n_j| \geq 2$ ; the sum  $S_2$  is taken over all  $x' \in I_{mN_\varepsilon}$  and  $y' \in I_{nN_\varepsilon}$  and  $m, n \in J$  such that  $m \neq n$  and  $\forall j = 1, \dots, d : |m_j - n_j| \leq 1$ . By Lemma 4.5, i) and condition **V2**, the sum  $S_1$  is estimated by

$$C(1 + \tau/\varepsilon)^{-d} (\tau/\varepsilon)^d \sum_{s \in \mathbb{Z}^d, |s| \geq N_\varepsilon} (1 + |s|)^{-\gamma},$$

which vanishes as  $\varepsilon \rightarrow 0$ , since  $N_\varepsilon \rightarrow +\infty$  and  $\gamma > d$ . To estimate the second sum  $S_2$  (the contribution of nearest neighbors  $I_{mN_\varepsilon}$  and  $I_{nN_\varepsilon}$ ) we choose a number  $p > d + 1$  and divide the sum onto two sums:  $S_2 = S_{21} + S_{22}$ , where the sum  $S_{21}$  is taken over all  $m \in J$  and  $x' \in I_{mN_\varepsilon}$ ,  $n \in \{n \in J : n \neq m, \forall j : |m_j - n_j| \leq 1\}$  and  $y' \in I_{nN_\varepsilon}$  such that  $|x' - y'| \geq N_\varepsilon^{1/p}$  and the second sum  $S_{22}$  is taken, respectively, over  $y'$  such that  $|x' - y'| \leq N_\varepsilon^{1/p}$ . The contribution of “non-boundary zones”  $S_{21}$  is

$$C(1 + \tau/\varepsilon)^{-d} (\tau/\varepsilon)^d \sum_{s \in \mathbb{Z}^d, |s| \geq N_\varepsilon^{1/p}} (1 + |s|)^{-\gamma}$$

which vanishes as  $\varepsilon \rightarrow 0$ . The contribution of “boundary zones”  $S_{22}$  is order of

$$C(1 + \tau/\varepsilon)^{-d} (\tau/\varepsilon N_\varepsilon)^d N_\varepsilon^{1/p + d - 1} N_\varepsilon^{d/p} \sim C N_\varepsilon^{(d+1)/p - 1}. \quad (4.21)$$

The number  $p$  is chosen such that  $(d + 1)/p - 1 < 0$ . Hence, (4.21) vanishes as  $\varepsilon \rightarrow 0$  by condition **V1**, ii). The decay (4.20) is proved.

*Step iii).* Now we can apply the condition **V1**, i) at the points  $[r/\varepsilon] - x', [r/\varepsilon] - y'$  of the same cube  $I_{[r/\varepsilon] - mN_\varepsilon}$  and obtain

$$|Q_\varepsilon([r/\varepsilon] - x', [r/\varepsilon] - y') - R(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, y' - x')| \leq C \min[(1 + |x' - y'|)^{-\gamma}, \varepsilon N_\varepsilon].$$

Then

$$Q_{\varepsilon, \tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon]) = \sum_{m \in J} \sum_{x', y' \in I_{mN_\varepsilon}} \mathcal{G}_{\tau/\varepsilon}^g(l + x') R(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, y' - x') \mathcal{G}_{\tau/\varepsilon}^g(y')^* + r_3(\varepsilon, \tau) + r_2(\varepsilon, \tau) + r_1(\varepsilon, \tau) + o(1). \quad (4.22)$$

Let us prove that  $\lim_{\varepsilon \rightarrow 0} r_3(\varepsilon, \tau) = 0$  for any  $\tau \in \mathbb{R}^1$ . Indeed, since for fixed  $x' \in I_{mN_\varepsilon}$  the sum  $\sum_{y' \in I_{mN_\varepsilon}} \min[(1 + |x' - y'|)^{-\gamma}, \varepsilon N_\varepsilon]$  is the order of  $(\varepsilon N_\varepsilon)^{1-d/\gamma}$ , we get, by Lemma 4.5, i),

$$\begin{aligned} |r_3(\varepsilon, \tau)| &\leq C \sum_{m \in J} \sum_{x', y' \in I_{mN_\varepsilon}} |\mathcal{G}_{\tau/\varepsilon}^g(l + x')| \min[(1 + |x' - y'|)^{-\gamma}, \varepsilon N_\varepsilon] |\mathcal{G}_{\tau/\varepsilon}^g(y')^*| \\ &\leq C(1 + \tau/\varepsilon)^{-d} (\tau/(\varepsilon N_\varepsilon))^d N_\varepsilon^d (\varepsilon N_\varepsilon)^{1-d/\gamma} \sim \varepsilon^{(1-\beta)(1-d/\gamma)} \rightarrow 0, \quad \varepsilon \rightarrow 0, \end{aligned}$$

by condition **V1** ii), since  $\beta < 1$  and  $\gamma > d$ .

*Step iv).* By similar arguments, as in *steps i)* and *ii)*, the sums in the RHS of (4.22) can be taken over  $\{y' \in \mathbb{Z}^d, m \in J, x' \in I_{mN_\varepsilon}\}$ . The sum in  $y'$  is a convolution which can be expressed by the product in the Fourier transform:

$$\begin{aligned} Q_{\varepsilon, \tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon]) &= (2\pi)^{-2d} \sum_{m \in J} \sum_{x' \in I_{mN_\varepsilon} \mathbb{T}^{2d}} \int e^{-i\theta \cdot l} e^{ix' \cdot (\theta' - \theta)} \hat{\mathcal{G}}_{\tau/\varepsilon}^g(\theta) \hat{R}(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, \theta') \\ &\quad \times \hat{\mathcal{G}}_{\tau/\varepsilon}^g(\theta')^* d\theta d\theta' + o_\tau(1) + o(1), \end{aligned}$$

where  $o_\tau(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  for any  $\tau \in \mathbb{R}^1 \setminus \{0\}$ . Further, since  $I_{mN_\varepsilon} = \{x' \in \mathbb{Z}^d : (m_j - 1/2)N_\varepsilon \leq x'_j < (m_j + 1/2)N_\varepsilon, j = 1, \dots, d\}$ , then

$$\sum_{x' \in I_{mN_\varepsilon}} e^{ix' \cdot (\theta' - \theta)} = \prod_{j=1}^d \frac{F(\theta'_j - \theta_j, N_\varepsilon, m_j)}{e^{i(\theta'_j - \theta_j)} - 1},$$

where  $F(\theta_j, N_\varepsilon, m_j) = e^{i\theta_j N_\varepsilon(m_j + 1/2)} - e^{i\theta_j N_\varepsilon(m_j - 1/2)}$ . Changing variables i)  $(\theta, \theta') \rightarrow (z, \theta')$ ,  $z = \theta' - \theta$  and ii)  $(z, \theta') \rightarrow (z, \theta)$ ,  $\theta = \theta'$ , one obtains

$$\begin{aligned} Q_{\varepsilon, \tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon]) &= (2\pi)^{-2d} \sum_{m \in J} \int_{[-\pi, \pi]^{2d}} e^{-i(\theta - z) \cdot l} \prod_{j=1}^d \frac{\alpha(z_j) F(z_j, N_\varepsilon, m_j)}{iz_j} \hat{\mathcal{G}}_{\tau/\varepsilon}^g(\theta - z) \\ &\quad \times \hat{R}(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, \theta) \hat{\mathcal{G}}_{\tau/\varepsilon}^g(\theta)^* d\theta dz + o_\tau(1), \end{aligned} \quad (4.23)$$

where  $\alpha(z) = \frac{iz}{e^{iz} - 1}$  if  $z \in (-\pi, \pi) \setminus 0$  and  $\alpha(0) = 1$ . Note that

$$\hat{\mathcal{G}}_t^g(\theta) = g(\theta) (\cos \Omega(\theta)t + \sin \Omega(\theta)t C(\theta)).$$

Hence, in the integrand in (4.23) we have for  $t = \tau/\varepsilon$ ,

$$\begin{aligned} &\hat{\mathcal{G}}_t^g(\theta - z) \hat{R}(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, \theta) \hat{\mathcal{G}}_t^g(\theta)^* \\ &= \sum_{\sigma, \sigma'=1}^s \Pi_\sigma(\theta - z) g(\theta - z) (\cos \omega_\sigma(\theta - z)t + \sin \omega_\sigma(\theta - z)t C_\sigma(\theta - z)) \\ &\quad \hat{R}(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, \theta) g(\theta) (\cos \omega_{\sigma'}(\theta)t + \sin \omega_{\sigma'}(\theta)t C_{\sigma'}^*(\theta)) \Pi_{\sigma'}(\theta), \end{aligned} \quad (4.24)$$

where  $C_\sigma(\theta) = \begin{pmatrix} 0 & 1/\omega_\sigma(\theta) \\ -\omega_\sigma(\theta) & 0 \end{pmatrix}$ . Let us consider one of the terms in (4.23). The proof for the remaining terms is similar,

$$\begin{aligned} I_\varepsilon &= (2\pi)^{-2d} \frac{1}{4} \int_{[-\pi, \pi]^d} e^{-i\theta \cdot l} e^{i\omega_{\sigma'}(\theta)\tau/\varepsilon} g(\theta) \sum_{m \in J} \left( \int_{-\pi}^{\pi} e^{iz_d l_d} \frac{\alpha(z_d) F(z_d, N_\varepsilon, m_d)}{iz_d} \dots \right. \\ &\quad \times \left( \int_{-\pi}^{\pi} e^{iz_2 l_2} \frac{\alpha(z_2) F(z_2, N_\varepsilon, m_2)}{iz_2} \left( \int_{-\pi}^{\pi} e^{iz_1 l_1} \frac{\alpha(z_1) F(z_1, N_\varepsilon, m_1)}{iz_1} e^{\pm i\omega_\sigma(\theta-z)\tau/\varepsilon} \right. \right. \\ &\quad \left. \left. \times g(\theta-z) \Pi_\sigma(\theta-z) \hat{R}(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, \theta) \Pi_{\sigma'}(\theta) dz_1 \right) dz_2 \right) \dots dz_d \Big) d\theta. \end{aligned} \quad (4.25)$$

Introduce  $\nu_1 = \nu_1(\theta_1, \theta_2 - z_2, \dots) = \pm[\nabla_1 \omega_\sigma(\theta_1, \theta_2 - z_2, \dots)\tau/(\varepsilon N_\varepsilon)]$ ,  $\nu_2 = \nu_2(\theta_1, \theta_2, \theta_3 - z_3, \dots) = \pm[\nabla_2 \omega_\sigma(\theta_1, \theta_2, \theta_3 - z_3, \dots)\tau/(\varepsilon N_\varepsilon)]$ , ...,  $\nu_d = \nu_d(\theta) = \pm[\nabla_d \omega_\sigma(\theta)\tau/(\varepsilon N_\varepsilon)]$ .

**Lemma 4.6** *Let condition I4 hold. Then*

$$\begin{aligned} I_\varepsilon &= (2\pi)^{-2d} \frac{1}{4} \int_{[-\pi, \pi]^d} e^{-i\theta \cdot l} e^{i\omega_{\sigma'}(\theta)\tau/\varepsilon} g(\theta) \left( \sum_{|m_d - \nu_d| \leq 2-\pi} \int_{-\pi}^{\pi} e^{iz_d l_d} \frac{\alpha(z_d) F(z_d, N_\varepsilon, m_d)}{iz_d} \dots \right. \\ &\quad \times \left( \sum_{|m_2 - \nu_2| \leq 2-\pi} \int_{-\pi}^{\pi} e^{iz_2 l_2} \frac{\alpha(z_2) F(z_2, N_\varepsilon, m_2)}{iz_2} \left( \sum_{|m_1 - \nu_1| \leq 2-\pi} \int_{-\pi}^{\pi} e^{iz_1 l_1} \frac{\alpha(z_1) F(z_1, N_\varepsilon, m_1)}{iz_1} g(\theta-z) \right. \right. \\ &\quad \left. \left. \times e^{\pm i\omega_\sigma(\theta-z)\tau/\varepsilon} \Pi_\sigma(\theta-z) \hat{R}(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, \theta) \Pi_{\sigma'}(\theta) dz_1 \right) dz_2 \right) \dots dz_d \Big) d\theta + o_\tau(1), \end{aligned} \quad (4.26)$$

where  $o_\tau(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  for any  $\tau \in \mathbb{R}^1$ .

**Proof.** We generalize the strategy of the proof of Proposition 3.6 from [2], where this assertion is proved for  $d = 1$ . The asymptotics (4.26) follows from (4.25) if we prove that the series over  $\max_j |m_j - \nu_j| \geq 3$  vanishes as  $\varepsilon \rightarrow 0$ .

First, let us consider the inner integral over  $z_1$  in (4.25) and denote it by  $I_\varepsilon(\theta, z', m)$ :

$$I_\varepsilon(\theta, z', m) = \int_{-\pi}^{\pi} a(\theta, z, m) \frac{e^{if_+(\theta, z, m_1)N_\varepsilon} - e^{if_-(\theta, z, m_1)N_\varepsilon}}{iz_1} dz_1,$$

where  $a(\theta, z, m) = \alpha(z_1)g(\theta-z)\Pi_\sigma(\theta-z)\hat{R}(\varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon, \theta)\Pi_{\sigma'}(\theta)$ ,  $z' = (z_2, \dots, z_d) \in [-\pi, \pi]^{d-1}$ ,  $\theta \in [-\pi, \pi]^d$  and

$$\begin{aligned} f_+(\theta, z, m_1) &= z_1 l_1 / N_\varepsilon + z_1(m_1 + 1/2) \pm \omega_\sigma(\theta - z)\tau/(\varepsilon N_\varepsilon), \\ f_-(\theta, z, m_1) &= z_1 l_1 / N_\varepsilon + z_1(m_1 - 1/2) \pm \omega_\sigma(\theta - z)\tau/(\varepsilon N_\varepsilon). \end{aligned}$$

We have  $f_+(\theta, z, m_1)|_{z_1=0} = f_-(\theta, z, m_1)|_{z_1=0} = \pm\omega_\sigma(\theta_1, \theta_2 - z_2, \dots)\tau/(\varepsilon N_\varepsilon)$ , and

$$\nabla_1 f_\pm(\theta, z, m_1)|_{z_1=0} = l_1 / N_\varepsilon + m_1 \pm 1/2 - \nu_1.$$

Hence,  $\nabla_1 f_{\pm}(\theta, z, m_1) \Big|_{z_1=0} \neq 0$  for  $|m_1 - \nu_1| \geq 3$ . Indeed, we can admit that  $|l_1/N_{\varepsilon}| \leq 1$  since  $N_{\varepsilon} \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , and the number  $l_1 \in \mathbb{Z}$  is fixed. Further, we apply to  $I_{\varepsilon}(\theta, z', m)$  the limit of Lemma 3.7 from [2],

$$\lim_{\varepsilon \rightarrow 0} \left[ I_{\varepsilon}(\theta, z', m) - \pi e^{\pm i \omega_{\sigma}(\theta_1, \theta_2 - z_2, \dots) \tau / \varepsilon} a(\theta, (0, z'), m) \right. \\ \left. \times \left( \operatorname{sgn} \nabla_1 f_{+}(\theta, z, m_1) \Big|_{z_1=0} - \operatorname{sgn} \nabla_1 f_{-}(\theta, z, m_1) \Big|_{z_1=0} \right) \right] = 0. \quad (4.27)$$

Moreover, we obtain that  $I_{\varepsilon}(\theta, z', m) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , uniformly in  $\theta \in [-\pi, \pi]^d$  and  $z' \in [-\pi, \pi]^{d-1}$ , since  $\operatorname{sgn} \nabla_1 f_{+}(\theta, z, m_1) \Big|_{z_1=0} = \operatorname{sgn} \nabla_1 f_{-}(\theta, z, m_1) \Big|_{z_1=0}$ . We proceed by induction for each inner integral over  $z_2, \dots, z_d$  and obtain that the integrals with  $\max_j |m_j - \nu_j| \geq 3$  vanish as  $\varepsilon \rightarrow 0$ . Further, we have to prove that the series over  $\max_j |m_j - \nu_j| \geq 3$  also vanish. This follows from two facts: i) the function  $a(\theta, z, m)$  has a structure of  $f(\theta, z) \hat{R}(\varepsilon[r/\varepsilon] - \varepsilon m N_{\varepsilon}, \theta)$  with a smooth function  $f$ , and ii)  $\hat{R}(r, \theta)$  satisfies condition **I4**. More exactly it is proved for the case  $d = 1$  in [2]. The proof admits generalization to the case  $d > 1$ , here we omit the detailed computations.

*Step v)* The next step is to prove that

$$I_{\varepsilon} = \frac{(2\pi)^{-2d}}{4} \int_{[-\pi, \pi]^d} e^{-i\theta \cdot l + i\omega_l(\theta) \tau / \varepsilon} g(\theta) \Pi_{\sigma}(\theta) \hat{R}(r \mp \nabla \omega_{\sigma'}(\theta) \tau, \theta) \Pi_{\sigma'}(\theta) \left( \int_{-\pi}^{\pi} e^{iz_d l_d + i\nu_d N_{\varepsilon} z_d} \alpha(z_d) \right. \\ \left. \frac{e^{i5/2 N_{\varepsilon} z_d} - e^{-i5/2 N_{\varepsilon} z_d}}{iz_d} \dots \left( \int_{-\pi}^{\pi} e^{iz_2 l_2 + i\nu_2 N_{\varepsilon} z_2} \alpha(z_2) \frac{e^{i5/2 N_{\varepsilon} z_2} - e^{-i5/2 N_{\varepsilon} z_2}}{iz_2} \right. \right. \\ \left. \left. \left( \int_{-\pi}^{\pi} e^{iz_1 l_1 + i\nu_1 N_{\varepsilon} z_1 \pm i\omega_{\sigma}(\theta - z) \tau / \varepsilon} \alpha(z_1) \frac{e^{i5/2 N_{\varepsilon} z_1} - e^{-i5/2 N_{\varepsilon} z_1}}{iz_1} g(\theta - z) dz_1 \right) dz_2 \right) \dots dz_d \right) d\theta \\ + o_{\tau}(1), \quad (4.28)$$

where  $o_{\tau}(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  for any  $\tau \in \mathbb{R}^1$ . It follows from (4.26) and the formula  $\sum_{|m_j - \nu_j| \leq 2} F(z_j, N_{\varepsilon}, m_j) = e^{i\nu_j N_{\varepsilon} z_j} (e^{i5/2 N_{\varepsilon} z_j} - e^{-i5/2 N_{\varepsilon} z_j})$ . Formula (4.28) is proved in Lemma 3.8 from [2] for the case  $d = 1$ . The proof is based on the condition **I4** for function  $\hat{R}$  and admits extension to the case  $d > 1$ .

Further, we apply (4.27) to the inner integrals from the RHS of (4.28) and obtain, for the inner integral over  $z_1$  (denote it by  $I_{\varepsilon}(\theta, z')$ , where  $z' = (z_2, \dots, z_d)$ ),

$$\lim_{\varepsilon \rightarrow 0} [I_{\varepsilon}(\theta, z') - 2\pi e^{\pm i \omega_{\sigma}(\theta_1, \theta_2 - z_2, \dots) \tau / \varepsilon} g(\theta_1, \theta_2 - z_2, \dots, \theta_d - z_d)] = 0, \quad (4.29)$$

since in this case  $I_{\varepsilon}(\theta, z') = \int_{\mathbb{T}^1} \alpha(z_1) g(\theta - z) (\exp(i f_{+}(\theta, z) N_{\varepsilon}) - \exp(i f_{-}(\theta, z) N_{\varepsilon})) / (iz_1) dz_1$  with

$$f_{+}(\theta, z) = z_1 l_1 / N_{\varepsilon} + z_1 5/2 + \nu_1 z_1 \pm \omega_{\sigma}(\theta - z) \tau / (\varepsilon N_{\varepsilon}), \\ f_{-}(\theta, z) = z_1 l_1 / N_{\varepsilon} - z_1 5/2 + \nu_1 z_1 \pm \omega_{\sigma}(\theta - z) \tau / (\varepsilon N_{\varepsilon}),$$

and  $\text{sgn} \nabla_1 f_{\pm}(\theta, z) \Big|_{z_1=0} = \text{sgn}(l_1/N_{\varepsilon} \pm 5/2) = \pm 1$  for fixed  $l_1 \in \mathbb{Z}$  and small enough  $\varepsilon > 0$ . Finally, we obtain

$$I_{\varepsilon} = \frac{(2\pi)^{-d}}{4} \int_{\mathbb{T}^d} e^{-i\theta \cdot l} e^{i(\omega_{\sigma'}(\theta) \pm \omega_{\sigma}(\theta))\tau/\varepsilon} g(\theta) \Pi_{\sigma}(\theta) \hat{R}(r \mp \nabla \omega_{\sigma}(\theta) \tau, \theta) \Pi_{\sigma'}(\theta) d\theta + o_{\tau}(1), \quad (4.30)$$

where  $o_{\tau}(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  for any  $\tau \neq 0$ .

*Step vi)* Note that the identities  $\omega_{\sigma}(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_{\pm}$  in the exponent (see (4.30)) with the  $\text{const}_{\pm} \neq 0$  are impossible by the condition **E5**. Furthermore, the oscillatory integrals with  $\omega_{\sigma}(\theta) \pm \omega_{\sigma'}(\theta) \not\equiv \text{const}_{\pm}$  vanish as  $\varepsilon \rightarrow 0$  by the condition **I1** and the Lebesgue-Riemann theorem. Hence, only the integrals with  $\omega_{\sigma}(\theta) - \omega_{\sigma'}(\theta) \equiv 0$  contribute to the integral (4.30) since  $\omega_{\sigma}(\theta) + \omega_{\sigma'}(\theta) \equiv 0$  would imply  $\omega_{\sigma}(\theta) \equiv \omega_{\sigma'}(\theta) \equiv 0$  which is impossible by **E4**. We return to formula (4.23) and applying (4.24) one obtains formulas (4.1).  $\blacksquare$

## 5 Proof of Theorems 3.4 and 3.5

### 5.1 Convergence of Wigner matrices

**Proof of Theorem 3.4.** Theorem 4.1 implies that for any  $r \in \mathbb{R}^d$ ,  $\tau \neq 0$  and  $y \in (2\mathbb{Z})^d$  the following convergence holds,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_{\tau/\varepsilon}^{\varepsilon} (a([r/\varepsilon] + y/2)^* \otimes a([r/\varepsilon] - y/2)) = \mathcal{W}^p(\tau; r, y), \quad (5.1)$$

where in the Fourier space one has

$$\begin{aligned} \hat{\mathcal{W}}^p(\tau; r, \theta) &= \frac{1}{2} \left( \Omega^{1/2} \hat{q}_{\tau, r}^{00}(\theta) \Omega^{1/2} + \Omega^{-1/2} \hat{q}_{\tau, r}^{11}(\theta) \Omega^{-1/2} \right. \\ &\quad \left. + i\Omega^{1/2} \hat{q}_{\tau, r}^{01}(\theta) \Omega^{-1/2} - i\Omega^{-1/2} \hat{q}_{\tau, r}^{10}(\theta) \Omega^{1/2} \right) \\ &= \mathcal{W}^p(\tau; r, \theta), \end{aligned} \quad (5.2)$$

by formulas (3.12), (3.13) and (4.1)–(4.3). Then convergence (3.14) follows from (5.1), (5.2) and Lemma 5.1.

**Lemma 5.1** *Let conditions **V2** and **E1** - **E3**, **E6** hold and  $\alpha < -d/2$ . Then*

$$\sup_{\varepsilon, t \in \mathbb{R}} \sup_{x, y \in \mathbb{Z}^d} \|Q_{\varepsilon, t}(x, y)\| \leq C < \infty. \quad (5.3)$$

**Proof** Applying (4.18) one has

$$Q_{\varepsilon, t}^{ij}(x, y) = \mathbb{E}_0^{\varepsilon} (Y^i(x, t) \otimes Y^j(y, t)) = \langle Q_{\varepsilon}(x', y'), \Phi_x^i(x', t) \otimes \Phi_y^j(y', t) \rangle,$$

where

$$\Phi_x^i(x', t) = (\mathcal{G}_t^{i0}(x - x'), \mathcal{G}_t^{i1}(x - x')), \quad x' \in \mathbb{Z}^d, \quad i = 0, 1.$$

Then the Parseval identity, (4.9) and condition **E6** imply

$$\|\Phi_x^i(\cdot, t)\|_2^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} |\hat{\Phi}_x^i(\theta, t)|^2 d\theta = (2\pi)^{-d} \int_{\mathbb{T}^d} (|\hat{\mathcal{G}}_t^{i0}(\theta)|^2 + |\hat{\mathcal{G}}_t^{i1}(\theta)|^2) d\theta \leq C_0 < \infty.$$

Then Corollary 4.4 gives

$$|Q_{\varepsilon, t}^{ij}(x, y)| = |\langle Q_\varepsilon(x', y'), \Phi_x^i(x', t) \otimes \Phi_y^j(y', t) \rangle| \leq C \|\Phi_x^i(\cdot, t)\|_2 \|\Phi_y^j(\cdot, t)\|_2 \leq C_1 < \infty,$$

where the constant  $C_1$  does not depend on  $x, y \in \mathbb{Z}^d$ ,  $t \in \mathbb{R}$  and  $\varepsilon > 0$ . ■

## 5.2 Weak convergence of measures $\mu_{\tau/\varepsilon, r}^\varepsilon$ as $\varepsilon \rightarrow 0$

Theorem 3.5 follows from Propositions 5.2 and 5.3. Proposition 5.2 ensures the existence of the limit measures of the family  $\{\mu_{\tau/\varepsilon, r}^\varepsilon, \varepsilon > 0\}$ , while Proposition 5.3 provides the uniqueness.

**Proposition 5.2** *Let conditions **V2** and **E1** - **E3**, **E6** hold. Then for any  $r \in \mathbb{R}^d$ ,  $\tau \neq 0$ , the family of measures  $\{\mu_{\tau/\varepsilon, r}^\varepsilon, \varepsilon > 0\}$  is weakly compact in  $\mathcal{H}_\alpha$  with any  $\alpha < -d/2$ , and the following bound holds,*

$$\sup_{\varepsilon \geq 0} \int \|Y_0\|_\alpha^2 \mu_{\tau/\varepsilon, r}^\varepsilon(dY_0) < \infty. \quad (5.4)$$

**Proof** Definition (2.1) implies

$$\begin{aligned} \int \|Y_0\|_\alpha^2 \mu_{\tau/\varepsilon, r}^\varepsilon(dY_0) &= \mathbb{E}_0^\varepsilon(\|T_{-[r/\varepsilon]}U(\tau/\varepsilon)Y_0\|_\alpha^2) \\ &= \sum_{x \in \mathbb{Z}^d} (1 + |x|^2)^\alpha \left( \text{tr } Q_{\varepsilon, \tau/\varepsilon}^{00}([r/\varepsilon] + x, [r/\varepsilon] + x) + \text{tr } Q_{\varepsilon, \tau/\varepsilon}^{11}([r/\varepsilon] + x, [r/\varepsilon] + x) \right). \end{aligned}$$

Since  $\alpha < -d/2$ , (5.4) follows from the bound (5.3). Now the compactness of the measures family  $\{\mu_t, t \in \mathbb{R}\}$  follows from the bound (5.4) by the Prokhorov Theorem [13, Lemma II.3.1] using the method of [13, Theorem XII.5.2], since the embedding  $\mathcal{H}_\alpha \subset \mathcal{H}_\beta$  is compact if  $\alpha > \beta$ . ■

Denote by  $\mathcal{Q}_{\tau, r}$  the quadratic form with the matrix kernel  $(q_{\tau, r}^{ij}(x - y))_{i, j=0, 1}$ ,

$$\mathcal{Q}_{\tau, r}(\Psi, \Psi) = \sum_{i, j=0, 1} \sum_{x, y \in \mathbb{Z}^d} (q_{\tau, r}^{ij}(x - y), \Psi^i(x) \otimes \Psi^j(y)), \quad \Psi \in \mathcal{D}. \quad (5.5)$$

**Proposition 5.3** *Let conditions **V1** - **V4** and **E1** - **E6** hold. Then for any  $r \in \mathbb{R}^d$ ,  $\tau \neq 0$  and  $\Psi \in \mathcal{D}$ ,*

$$\lim_{\varepsilon \rightarrow 0} \int \exp(i\langle Y, \Psi \rangle) \mu_{\tau/\varepsilon, r}^\varepsilon(dY) = \exp \left\{ -\frac{1}{2} \mathcal{Q}_{\tau, r}(\Psi, \Psi) \right\}. \quad (5.6)$$

Proposition 5.3 is proved in Sections 6 - 9.

## 6 Convergence of characteristic functionals

To prove Theorem 3.5, it remains to check Proposition 5.3. Let us rewrite (5.6) as

$$\hat{\mu}_{\tau/\varepsilon, r}^\varepsilon(\Psi) = \mathbb{E}_0^\varepsilon(\exp\{i\langle T_{-[r/\varepsilon]}U(\tau/\varepsilon)Y_0, \Psi \rangle\}) \rightarrow \hat{\mu}_{\tau, r}^G(\Psi), \quad \varepsilon \rightarrow 0. \quad (6.1)$$

We will prove it in Sections 8, 9. In this section we evaluate  $\langle T_{-[r/\varepsilon]}U(t)Y_0, \Psi \rangle$ ,  $t \in \mathbb{R}$ , by using the following duality arguments.

### 6.1 Duality arguments

Remember that  $Y_0 \in \mathcal{H}_\alpha$  with  $\alpha < -d/2$ . For  $t \in \mathbb{R}$  introduce a ‘formal adjoint’ operator  $U'(t)$  from space  $\mathcal{D}$  to  $\mathcal{H}_{-\alpha}$ :

$$\langle Y, U'(t)\Psi \rangle = \langle U(t)Y, \Psi \rangle, \quad \Psi \in \mathcal{D}, \quad Y \in \mathcal{H}_\alpha. \quad (6.2)$$

Let us denote by  $\Phi_r(\cdot, t) = U'(t)T_{[r/\varepsilon]}\Psi$ . Then using (6.2) we obtain

$$\langle T_{-[r/\varepsilon]}U(t)Y_0, \Psi \rangle = \langle Y_0, \Phi_r(\cdot, t) \rangle, \quad t \in \mathbb{R}, \quad \varepsilon > 0, \quad r \in \mathbb{R}^d. \quad (6.3)$$

The adjoint group  $U'(t)$  admits the following convenient description. Lemma 6.1 below displays that the action of group  $U'(t)$  coincides with the action of  $U(t)$ , up to the order of the components.

**Lemma 6.1** *For  $\Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$  we have*

$$\Phi(\cdot, t) = U'(t)\Psi = (\dot{\psi}(\cdot, t), \psi(\cdot, t)), \quad (6.4)$$

where  $\psi(x, t)$  is the solution of Eqn (2.1) with the initial data  $(u_0, v_0) = (\Psi^1, \Psi^0)$ .

The lemma allows us to construct the oscillatory integral representation for  $\Phi_r(x, t)$ . Namely, (6.4) implies that in Fourier representation for  $\Phi_r(\cdot, t) = U'(t)T_{[r/\varepsilon]}\Psi$  we have

$$\dot{\hat{\Phi}}_r(\theta, t) = \hat{\mathcal{A}}^*(\theta)\hat{\Phi}_r(\theta, t), \quad \hat{\Phi}(t, t) = \hat{\mathcal{G}}_t^*(\theta)e^{i[r/\varepsilon]\cdot\theta}\hat{\Psi}(\theta),$$

where

$$\hat{\mathcal{A}}^*(\theta) = \begin{pmatrix} 0 & -\hat{V}(\theta) \\ 1 & 0 \end{pmatrix}, \quad \hat{\mathcal{G}}_t^*(\theta) = \begin{pmatrix} \cos \Omega(\theta)t & -\Omega(\theta) \sin \Omega(\theta)t \\ \Omega^{-1}(\theta) \sin \Omega(\theta)t & \cos \Omega(\theta)t \end{pmatrix}.$$

Therefore

$$\Phi_r(x, t) = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-i\theta \cdot x} \hat{\mathcal{G}}_t^*(\theta) e^{i[r/\varepsilon]\cdot\theta} \hat{\Psi}(\theta) d\theta, \quad x \in \mathbb{Z}^d. \quad (6.5)$$

**Definition 6.2**  $\mathcal{D}^0 = \{\Psi \in \mathcal{D} : \hat{\Psi}(\theta) = 0 \text{ in a neighborhood of } \mathcal{C}\}$ .

From (6.5) we obtain

**Lemma 6.3** *For any fixed  $\Psi \in \mathcal{D}^0$  the following bounds hold:*

- i)  $|\Phi_r(x, t)| \leq C t^{-d/2}$ ,  $x \in \mathbb{Z}^d$ .
- ii) For any  $p > 0$  there exist  $C_p, \gamma_g > 0$  such that  $|\Phi_r(x, t)| \leq C_p(|t| + |x| + 1)^{-p}$ ,  $|x| \geq \gamma_g t$ .

This lemma follows from Lemma 4.5 and the definition of  $\mathcal{D}^0$ .

## 6.2 Equicontinuity of characteristic functionals

Let us show that we can restrict ourselves to  $\Psi \in \mathcal{D}^0$ .

**Lemma 6.4** *The convergence (6.1) it suffices to prove for  $\Psi \in \mathcal{D}^0$  only.*

**Proof.** *Step i).* For simplicity, let us put  $t = \tau/\varepsilon$ . Denote by

$$\mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi) = \int |\langle Y_0, \Psi \rangle|^2 d\mu_{t,r}^\varepsilon(dY_0).$$

We prove at first that

$$\sup_{\varepsilon > 0, t \in \mathbb{R}, r \in \mathbb{R}^d} |\mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi)| \leq C \|\Psi\|_2^2, \quad \Psi \in \mathcal{D}. \quad (6.6)$$

Indeed, by (6.3) we have

$$\mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi) = \mathbb{E}_0^\varepsilon(|\langle T_{-[r/\varepsilon]}U(t)Y_0, \Psi \rangle|^2) = \langle Q_\varepsilon(x, y), \Phi_r(x, t) \otimes \Phi_r(y, t) \rangle.$$

So, by Corollary 4.4 we obtain

$$\sup_{\varepsilon > 0, t \in \mathbb{R}, r \in \mathbb{R}^d} |\mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi)| \leq C \sup_{t \in \mathbb{R}, r \in \mathbb{R}^d} \|\Phi_r(\cdot, t)\|_2^2.$$

Finally, by the Parseval identity and condition **E6**, we get

$$\|\Phi_r(\cdot, t)\|_2^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} \|\mathcal{G}_t^*(\theta)\| |\hat{\Psi}(\theta)|^2 d\theta \leq C \|\Psi\|_2^2.$$

The bound (6.6) is proved.

*Step ii).* By the Cauchy-Schwarz inequality,

$$\begin{aligned} |\hat{\mu}_{t,r}^\varepsilon(\Psi_1) - \hat{\mu}_{t,r}^\varepsilon(\Psi_2)| &= \left| \int \left( e^{i\langle Y, \Psi_1 \rangle} - e^{i\langle Y, \Psi_2 \rangle} \right) \mu_{t,r}^\varepsilon(dY) \right| \leq \int \left| e^{i\langle Y, \Psi_1 - \Psi_2 \rangle} - 1 \right| \mu_{t,r}^\varepsilon(dY) \\ &\leq \int |\langle Y, \Psi_1 - \Psi_2 \rangle| \mu_{t,r}^\varepsilon(dY) \leq \left( \int |\langle Y, \Psi_1 - \Psi_2 \rangle|^2 \mu_{t,r}^\varepsilon(dY) \right)^{1/2} \\ &= (\mathcal{Q}_{\varepsilon,t,r}(\Psi_1 - \Psi_2, \Psi_1 - \Psi_2))^{1/2} \leq C \|\Psi_1 - \Psi_2\|_2, \end{aligned}$$

where a constant  $C$  does not depend on  $\varepsilon > 0$ ,  $t \in \mathbb{R}^1$  and  $r \in \mathbb{R}^d$ . Hence, the characteristic functionals  $\hat{\mu}_{t/\varepsilon,r}^\varepsilon(\Psi)$ ,  $t \in \mathbb{R}$ ,  $\varepsilon > 0$ ,  $r \in \mathbb{R}^d$ , are equicontinuous in the space  $\mathcal{D}$  endowed with the norm  $\ell^2$ . In the turn, the set  $\mathcal{D}^0$  is dense in this space.  $\blacksquare$

## 7 Bernstein's 'rooms-corridors' partition

Let us introduce a 'room-corridor' partition of the ball  $\{x \in \mathbb{Z}^d : |x| \leq \gamma_g t\}$  with  $\gamma_g$  from Lemma 6.3 ii). For  $t > 0$  we choose below  $\Delta_t, \rho_t \in \mathbb{N}$  (we will specify the asymptotical relations between  $t$ ,  $\Delta_t$  and  $\rho_t$ ). Let us set  $h_t = \Delta_t + \rho_t$  and

$$a^j = jh_t, \quad b^j = a^j + \Delta_t, \quad j \in \mathbb{Z}, \quad n_t = [\gamma_g t / h_t]. \quad (7.1)$$

We call the slabs  $R_t^j = \{x \in \mathbb{Z}^d, |x| \leq n_t h_t : a^j \leq x_d < b^j\}$  the 'rooms',  $C_t^j = \{x \in \mathbb{Z}^d, |x| \leq n_t h_t : b^j \leq x_d < a^{j+1}\}$  the 'corridors' and  $L_t = \{x \in \mathbb{Z}^d, |x| > n_t h_t\}$  the 'tails'. Here  $x = (x_1, \dots, x_d)$ ,  $\Delta_t$  is the width of a room, and  $\rho_t$  is that of a corridor. Let us denote by  $\chi_t^j$  the indicator of the room  $R_t^j$ ,  $\xi_t^j$  that of the corridor  $C_t^j$ , and  $\eta_t$  that of the tail  $L_t$ . Then

$$\sum_j [\chi_t^j(x) + \xi_t^j(x)] + \eta_t(x) = 1, \quad x \in \mathbb{Z}^d,$$

where the sum  $\sum_j$  stands for  $\sum_{j=-n_t}^{n_t-1}$ . Hence we get the following Bernstein's type representation:

$$\langle Y_0, \Phi_r(\cdot, t) \rangle = \sum_j [\langle Y_0, \chi_t^j \Phi_r(\cdot, t) \rangle + \langle Y_0, \xi_t^j \Phi_r(\cdot, t) \rangle] + \langle Y_0, \eta_t \Phi_r(\cdot, t) \rangle. \quad (7.2)$$

Let us introduce the random variables  $r_t^j$ ,  $c_t^j$ ,  $l_t$  by

$$r_t^j = \langle Y_0, \chi_t^j \Phi_r(\cdot, t) \rangle, \quad c_t^j = \langle Y_0, \xi_t^j \Phi_r(\cdot, t) \rangle, \quad l_t = \langle Y_0, \eta_t \Phi_r(\cdot, t) \rangle. \quad (7.3)$$

Then (7.2) becomes

$$\langle Y_0, \Phi_r(\cdot, t) \rangle = \sum_j (r_t^j + c_t^j) + l_t. \quad (7.4)$$

**Lemma 7.1** *Let conditions **V1** - **V2** hold and  $\Psi \in \mathcal{D}^0$ . The following bounds hold for  $t > 1$ :*

$$\mathbb{E}_0^\varepsilon(|r_t^j|^2) \leq C(\Psi) \Delta_t / t, \quad \forall j, \quad (7.5)$$

$$\mathbb{E}_0^\varepsilon(|c_t^j|^2) \leq C(\Psi) \rho_t / t, \quad \forall j, \quad (7.6)$$

$$\mathbb{E}_0^\varepsilon(|l_t|^2) \leq C_p(\Psi) (1+t)^{-p}, \quad \forall p > 0. \quad (7.7)$$

**Proof** The bound (7.7) follows from (4.17). We discuss (7.5), and (7.6) can be done in a similar way. Let us express  $\mathbb{E}_0^\varepsilon(|r_t^j|^2)$  in the correlation matrices. Definition (7.3) implies by the Fubini Theorem that

$$\mathbb{E}_0^\varepsilon(|r_t^j|^2) = \langle Q_\varepsilon(x, y), \chi_t^j(x) \Phi_r(x, t) \otimes \chi_t^j(y) \Phi_r(y, t) \rangle. \quad (7.8)$$

According to (4.17) and (4.16), Eqn (7.8) implies that

$$\begin{aligned} \mathbb{E}_0^\varepsilon(|r_t^j|^2) &\leq C t^{-d} \sum_{x, y} \chi_t^j(x) \|Q_\varepsilon(x, y)\| \\ &= C t^{-d} \sum_x \chi_t^j(x) \sum_z \|Q_\varepsilon(x, y)\| \leq C \Delta_t / t, \end{aligned} \quad (7.9)$$

where  $\|Q_\varepsilon(x, y)\|$  stands for the norm of a matrix  $(Q_\varepsilon^{ij}(x, y))$ . Therefore, (7.9) follows from Lemma 4.3. ■

## 8 Ibragimov-Linnik Central Limit Theorem

In this section we prove the convergence (6.1). As was said, we use a version of the Central Limit Theorem developed by Ibragimov and Linnik. If  $\mathcal{Q}_{\tau,r}(\Psi, \Psi) = 0$ , the convergence (6.1) is obvious. Indeed, then,

$$\begin{aligned} & \left| \mathbb{E}_0^\varepsilon \left( \exp \{ i \langle Y_0, \Phi_r(\cdot, \tau/\varepsilon) \rangle \} \right) - \hat{\mu}_{\tau,r}^G(\Psi) \right| = \mathbb{E}_0^\varepsilon (| \exp \{ i \langle Y_0, \Phi_r(\cdot, \tau/\varepsilon) \rangle \} - 1 |) \\ & \leq \mathbb{E}_0^\varepsilon (| \langle Y_0, \Phi_r(\cdot, \tau/\varepsilon) \rangle |) \leq \left( \mathbb{E}_0^\varepsilon (| \langle Y_0, \Phi_r(\cdot, \tau/\varepsilon) \rangle |^2) \right)^{1/2} \\ & = \left( \langle Q_\varepsilon(x, y), \Phi_r(x, \tau/\varepsilon) \otimes \Phi_r(y, \tau/\varepsilon) \rangle \right)^{1/2} = \left( \mathcal{Q}_{\varepsilon, \tau/\varepsilon, r}(\Psi, \Psi) \right)^{1/2}, \end{aligned} \quad (8.1)$$

where  $\mathcal{Q}_{\varepsilon, \tau/\varepsilon, r}(\Psi, \Psi) \rightarrow \mathcal{Q}_{\tau, r}(\Psi, \Psi) = 0$ ,  $\varepsilon \rightarrow 0$ . Therefore, (6.1) follows from Theorem 3.4. Thus, we may assume that for a given  $\Psi \in \mathcal{D}^0$ ,

$$\mathcal{Q}_{\tau, r}(\Psi, \Psi) \neq 0. \quad (8.2)$$

Let us choose  $0 < \delta < 1$  and

$$\rho_t \sim t^{1-\delta}, \quad \Delta_t \sim \frac{t}{\log t}, \quad t \rightarrow \infty. \quad (8.3)$$

**Lemma 8.1** *The following limit holds,*

$$n_t \left[ \left( \frac{\rho_t}{t} \right)^{1/2} + (1 + \rho_t)^{-\kappa} \right] + n_t^2 \frac{\rho_t}{t} \rightarrow 0, \quad t \rightarrow \infty, \quad (8.4)$$

where a constant  $\kappa > 0$ .

Indeed, (8.3) implies that  $h_t = \rho_t + \Delta_t \sim \frac{t}{\log t}$ ,  $t \rightarrow \infty$ . Therefore,  $n_t \sim \frac{t}{h_t} \sim \log t$ . Then (8.4) follows by (8.3).  $\blacksquare$

For simplicity, we put  $t = \tau/\varepsilon$ . By the triangle inequality,

$$\begin{aligned} \left| \mathbb{E}_0^\varepsilon \left( \exp \{ i \langle Y_0, \Phi_r(\cdot, t) \rangle \} \right) - \hat{\mu}_{\tau,r}^G(\Psi) \right| & \leq \left| \mathbb{E}_0^\varepsilon \left( \exp \{ i \langle Y_0, \Phi_r(\cdot, t) \rangle \} \right) - \mathbb{E}_0^\varepsilon \left( \exp \{ i \sum_j r_t^j \} \right) \right| \\ & \quad + \left| \exp \left\{ -\frac{1}{2} \sum_j \mathbb{E}_0^\varepsilon (|r_t^j|^2) \right\} - \exp \left\{ -\frac{1}{2} \mathcal{Q}_{\tau,r}(\Psi, \Psi) \right\} \right| \\ & \quad + \left| \mathbb{E}_0^\varepsilon \left( \exp \{ i \sum_j r_t^j \} \right) - \exp \left\{ -\frac{1}{2} \sum_j \mathbb{E}_0^\varepsilon (|r_t^j|^2) \right\} \right| \\ & = I_1 + I_2 + I_3. \end{aligned} \quad (8.5)$$

We are going to show that all summands  $I_1$ ,  $I_2$ ,  $I_3$  tend to zero as  $t \rightarrow \infty$ .

*Step (i)* Eqn (7.4) implies

$$\begin{aligned} I_1 & = \left| \mathbb{E}_0^\varepsilon \left( \exp \{ i \sum_j r_t^j \} \left( \exp \{ i \sum_j c_t^j + i l_t \} - 1 \right) \right) \right| \\ & \leq \sum_j \mathbb{E}_0^\varepsilon (|c_t^j|) + \mathbb{E}_0^\varepsilon (|l_t|) \leq \sum_j \left( \mathbb{E}_0^\varepsilon (|c_t^j|^2) \right)^{1/2} + \left( \mathbb{E}_0^\varepsilon (|l_t|^2) \right)^{1/2}. \end{aligned} \quad (8.6)$$

From (8.6), (7.6), (7.7) and (8.4) we obtain that

$$I_1 \leq Cn_t(\rho_t/t)^{1/2} + C_p t^{-p} \rightarrow 0, \quad t \rightarrow \infty.$$

*Step (ii)* By the triangle inequality,

$$\begin{aligned} I_2 &\leq \frac{1}{2} \left| \sum_j \mathbb{E}_0^\varepsilon(|r_t^j|^2) - \mathcal{Q}_{\tau,r}(\Psi, \Psi) \right| \leq \frac{1}{2} \left| \mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi) - \mathcal{Q}_{\tau,r}(\Psi, \Psi) \right| \\ &\quad + \frac{1}{2} \left| \mathbb{E}_0^\varepsilon((\sum_j r_t^j)^2) - \sum_j \mathbb{E}_0^\varepsilon(|r_t^j|^2) \right| + \frac{1}{2} \left| \mathbb{E}_0^\varepsilon((\sum_j r_t^j)^2) - \mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi) \right| \\ &= I_{21} + I_{22} + I_{23}, \end{aligned} \quad (8.7)$$

where  $\mathcal{Q}_{\varepsilon,t,r}$  is the quadratic form with the matrix kernel  $Q_{\varepsilon,t,r}^{ij}(x, y)$ . Theorem 3.4 implies that  $I_{21}|_{t=\tau/\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . As for  $I_{22}$ , we first obtain that

$$I_{22} \leq \sum_{j < l} |\mathbb{E}_0^\varepsilon(r_t^j r_t^l)|. \quad (8.8)$$

The distance between the different rooms  $R_t^j$  is greater or equal to  $\rho_t$  according to (7.1). Then, by Lemma 6.3, i) and condition **V2**,

$$\begin{aligned} I_{22} &\leq \sum_{j < l} |\langle Q_\varepsilon(x, y), \chi_t^j \Phi_r(x, t) \otimes \chi_t^l \Phi_r(y, t) \rangle| \\ &\leq Ct^{-d} \sum_{j < l} \sum_x \chi_t^j(x) \sum_y \chi_t^l(y) (1 + |x - y|)^{-\gamma} \\ &\sim t^{-d} n_t^2 t^{d-1} \Delta_t \int_{\rho_t}^{+\infty} (1 + s)^{-\gamma} s^{d-1} ds \sim n_t (1 + \rho_t)^{-\gamma+d}, \end{aligned} \quad (8.9)$$

which vanishes as  $t \rightarrow \infty$  because of (8.4) and  $\gamma > d$ . Finally, it remains to check that  $I_{23} \rightarrow 0$ ,  $t \rightarrow \infty$ . We have

$$\mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi) = \mathbb{E}_0^\varepsilon(\langle Y_0, \Phi_r(\cdot, t) \rangle^2) = \mathbb{E}_0^\varepsilon((\sum_j (r_t^j + c_t^j) + l_t)^2),$$

according to (7.4). Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned} I_{23} &\leq \left| \mathbb{E}_0^\varepsilon((\sum_j r_t^j)^2) - \mathbb{E}_0^\varepsilon((\sum_j r_t^j + \sum_j c_t^j + l_t)^2) \right| \\ &\leq Cn_t \sum_j \mathbb{E}_0^\varepsilon(|c_t^j|^2) + C_1 \left( \mathbb{E}_0^\varepsilon((\sum_j r_t^j)^2) \right)^{1/2} \left( n_t \sum_j \mathbb{E}_0^\varepsilon(|c_t^j|^2) + \mathbb{E}_0^\varepsilon(|l_t|^2) \right)^{1/2} \\ &\quad + C \mathbb{E}_0^\varepsilon(|l_t|^2). \end{aligned} \quad (8.10)$$

Then (7.5), (8.8) and (8.9) imply

$$\begin{aligned} \mathbb{E}_0^\varepsilon((\sum_j r_t^j)^2) &\leq \sum_j \mathbb{E}_0^\varepsilon(|r_t^j|^2) + 2 \sum_{j < l} |\mathbb{E}_0^\varepsilon(r_t^j r_t^l)| \\ &\leq Cn_t \Delta_t / t + C_1 n_t (1 + \rho_t)^{-\gamma+d} \leq C_2 < \infty. \end{aligned}$$

Now (7.6), (7.7), (8.10) and (8.4) yield

$$I_{23} \leq C_1 n_t^2 \rho_t / t + C_2 n_t (\rho_t / t)^{1/2} + C_3 t^{-p} \rightarrow 0, \quad t \rightarrow \infty.$$

So, the terms  $I_{21}$ ,  $I_{22}$ ,  $I_{23}$  in (8.7) tend to zero. Then (8.7) implies that for  $t = \tau/\varepsilon$

$$I_2 \leq \frac{1}{2} \left| \sum_j \mathbb{E}_0^\varepsilon(|r_t^j|^2) - \mathcal{Q}_{\tau,r}(\Psi, \Psi) \right| \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (8.11)$$

*Step (iii)* It remains to verify that for  $t = \tau/\varepsilon$

$$I_3 = \left| \mathbb{E}_0^\varepsilon(\exp\{i \sum_j r_t^j\}) - \exp\left\{-\frac{1}{2} \sum_j \mathbb{E}_0^\varepsilon(|r_t^j|^2)\right\} \right| \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Condition **V3** yields

$$\begin{aligned} & \left| \mathbb{E}_0^\varepsilon(\exp\{i \sum_j r_t^j\}) - \prod_{-n_t}^{n_t-1} \mathbb{E}_0^\varepsilon(\exp\{ir_t^j\}) \right| \\ & \leq \left| \mathbb{E}_0^\varepsilon(\exp\{ir_t^{-n_t}\} \exp\{i \sum_{-n_t+1}^{n_t-1} r_t^j\}) - \mathbb{E}_0^\varepsilon(\exp\{ir_t^{-n_t}\}) \mathbb{E}_0^\varepsilon(\exp\{i \sum_{-n_t+1}^{n_t-1} r_t^j\}) \right| \\ & \quad + \left| \mathbb{E}_0^\varepsilon(\exp\{ir_t^{-n_t}\}) \mathbb{E}_0^\varepsilon(\exp\{i \sum_{-n_t+1}^{n_t-1} r_t^j\}) - \prod_{-n_t}^{n_t-1} \mathbb{E}_0^\varepsilon(\exp\{ir_t^j\}) \right| \\ & \leq C(1 + \rho_t)^{-\kappa} + \left| \mathbb{E}_0^\varepsilon(\exp\{i \sum_{-n_t+1}^{n_t-1} r_t^j\}) - \prod_{-n_t+1}^{n_t-1} \mathbb{E}_0^\varepsilon(\exp\{ir_t^j\}) \right|. \end{aligned}$$

We then apply condition **V3** recursively and obtain, according to Lemma 8.1,

$$\left| \mathbb{E}_0^\varepsilon(\exp\{i \sum_j r_t^j\}) - \prod_{-n_t}^{n_t-1} \mathbb{E}_0^\varepsilon(\exp\{ir_t^j\}) \right| \leq C n_t (1 + \rho_t)^{-\kappa} \Big|_{t=\tau/\varepsilon} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

It remains to check that for  $t = \tau/\varepsilon$

$$\left| \prod_{-n_t}^{n_t-1} \mathbb{E}_0^\varepsilon(\exp\{ir_t^j\}) - \exp\left\{-\frac{1}{2} \sum_j \mathbb{E}_0^\varepsilon(|r_t^j|^2)\right\} \right| \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

According to the standard statement of the Central Limit Theorem (see, e.g., [9, Theorem 4.7]), it suffices to verify the Lindeberg condition:  $\forall \delta > 0$ ,

$$\frac{1}{\sigma_t} \sum_j \mathbb{E}_0^{\varepsilon, \delta \sqrt{\sigma_t}}(|r_t^j|^2) \Big|_{t=\tau/\varepsilon} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Here  $\sigma_t = \sum_j \mathbb{E}_0^\varepsilon(|r_t^j|^2)$ , and  $\mathbb{E}_0^{\varepsilon, a}(f) \equiv \mathbb{E}_0^\varepsilon(X^a f)$ , where  $X^a$  is the indicator of the event  $|f| > a^2$ . Note that (8.11) and (8.2) imply that  $\sigma_{\tau/\varepsilon} \rightarrow \mathcal{Q}_{\tau,r}(\Psi, \Psi) \neq 0$ ,  $\varepsilon \rightarrow 0$ . Hence it remains to verify that

$$\sum_j \mathbb{E}_0^{\varepsilon, a}(|r_{\tau/\varepsilon}^j|^2) \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad \text{for any } a > 0. \quad (8.12)$$

We check Eqn (8.12) in Section 9. This will complete the proof of Proposition 5.3. ■

## 9 The Lindeberg condition

The proof of (8.12) is reduced to the proof of the following convergence

$$\sum_j \mathbb{E}_0^\varepsilon(|r_{\tau/\varepsilon}^j|^4) \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad (9.1)$$

by using Chebyshev's inequality. We deduce (9.1) from the following lemma.

**Lemma 9.1** *Let the conditions of Theorem 3.5 hold. Then for any  $\Psi \in \mathcal{D}^0$  the following bounds hold,*

$$\mathbb{E}_0^\varepsilon(|r_t^j|^4) \leq C(\Psi)\Delta_t^2/t^2, \quad t > 1. \quad (9.2)$$

**Proof.** *Step 1* Given four points  $x^1, x^2, x^3, x^4 \in \mathbb{Z}^d$ , we set  $M_\varepsilon^{(4)}(x^1, \dots, x^4) = \mathbb{E}_0^\varepsilon(Y(x^1) \otimes \dots \otimes Y(x^4))$ . Then, similarly to (7.8) we have

$$\mathbb{E}_0^\varepsilon(|r_t^j|^4) = \langle \chi_t^j(x^1) \dots \chi_t^j(x^4) M_\varepsilon^{(4)}(x^1, \dots, x^4), \Phi_r(x^1, t) \otimes \dots \otimes \Phi_r(x^4, t) \rangle. \quad (9.3)$$

Lemma 6.3, i) implies

$$\mathbb{E}_0^\varepsilon(|r_t^j|^4) \leq C t^{-2d} \sum_{i=2}^4 \sum_{\bar{x} \in (\mathbb{Z}^d)^4} \chi_t^j(x^1) \dots \chi_t^j(x^4) |M_\varepsilon^{(4)}(\bar{x})|. \quad (9.4)$$

By condition **V4**, we have

$$\sum_{\bar{x} \in (\mathbb{Z}^d)^4} \chi_t^j(x^1) \dots \chi_t^j(x^4) |M_\varepsilon^{(4)}(\bar{x})| \leq \sum_{(i_1, i_2, i_3, i_4) \in P\{1, 2, 3, 4\}} V_{i_1, i_2, i_3, i_4}(t), \quad (9.5)$$

where

$$V_{i_1, i_2, i_3, i_4}(t) = C \sum_{\bar{x}} \chi_t^j(x^1) \dots \chi_t^j(x^4) (1 + |x^{i_1} - x^{i_2}|)^{-\gamma} (1 + |x^{i_3} - x^{i_4}|)^{-\gamma}.$$

Similarly to (7.8), we have

$$\begin{aligned} V_{i_1, i_2, i_3, i_4}(t) &\leq C \sum_{x^{i_1}} \chi_t^j(x^{i_1}) \sum_{x^{i_2}} \chi_t^j(x^{i_2}) (1 + |x^{i_1} - x^{i_2}|)^{-\gamma} \\ &\quad \times \sum_{x^{i_3}} \chi_t^j(x^{i_3}) \sum_{x^{i_4}} \chi_t^j(x^{i_4}) (1 + |x^{i_3} - x^{i_4}|)^{-\gamma} \\ &\sim \left[ \Delta_t t^{d-1} \sum_{x^{i_2}} (1 + |x^{i_1} - x^{i_2}|)^{-\gamma} \right]^2. \end{aligned}$$

The sum in  $x^{i_2}$  is bounded since  $\gamma > d$ . Hence,

$$V_{i_1, i_2, i_3, i_4}(t) \leq C \Delta_t^2 t^{2d-2}. \quad (9.6)$$

Now the estimate (9.2) follows from (9.4), (9.5) and (9.6). This completes the proof of Lemma 9.1.  $\blacksquare$

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